

11/15/69
HP

ON ABSORPTION OF SOUND IN DIELECTRIC SOLIDS

MILAN OŽVOLD, Bratislava

This paper deals with the theoretical investigation of the sound wave absorption in dielectrics caused by anharmonic interaction of the sound with thermal phonons. The absorption coefficient and the effective elastic constants are determined from the solution of the wave equation in the quasi-harmonic approximation and the nonequilibrium gas of the phonons is described by the Boltzmann equation.

INTRODUCTION

In the present paper the sound absorption on thermal phonons in dielectric crystals is theoretically investigated. The sound wave, propagating through the crystal, interacts with the thermal phonons through the anharmonicity of the interatomic forces. Thermal phonons are also subjected to this interaction with one another and therefore have the final life time τ . If the sound wave frequency Ω is higher than the frequency of collisions of the thermal phonons ($\Omega\tau > 1$), this interaction can be investigated as a collision of sound wave phonons with individual thermal phonons [1] (see also [2]). However, in the case of the sound wave, except for the low temperatures and high frequencies, this condition is not fulfilled. In the reverse case, when $\Omega\tau < 1$, we can consider the interaction of the sound wave not with individual thermal phonons, but with the entire assembly of thermal phonons. The case when $\Omega\tau \sim 1$ must usually be treated as an interpolation between the two limits. The extrapolation from the region of the infinite life time of thermal phonons into the region of finite τ was made by Simons [3] and further used by Maris [4]; these calculations are a modification of the quantummechanical approach of Landau and Rumer [1]. The theory for the calculation of the absorption coefficient in the region $\Omega\tau < 1$ was suggested by Akhiezer [7], and extrapolated in a simple way by Bönnel and Dransfeld [6]. An improved version of this approach has been given by Woodruff and Ehrenreich [5]. In this paper the problem of sound propagation, its absorption and the thermal conductivity in dielectric crystals are studied from a uniform point

of view. Our method is based on the solution of the Boltzmann equation for a phonon gas and the elastic wave equation in the quasi-harmonic approximation. The term expressing the interaction of the acoustic wave with a phonon gas enters the wave equation and in this way determines not only the sound absorption, but also the elastic constant change. In the mentioned works, the problem of elastic constants was not solved. Leibfried and Ludwig [13] showed that the sound propagation at high temperatures is described by adiabatic elastic constants. Our method is not limited to high temperatures and enables us to obtain the elastic constant dependence on ΩT . A contribution to this problem has recently been made also by Götze and Michel [16]. They started from the same system of equations as we did. The principal difference between their work and this one consists in the method of the Boltzmann equation solution, further their results are very implicit and there is little space devoted to the sound absorption; besides, they do not cover the whole range of ΩT .

THE WAVE EQUATION IN QUASI-HARMONIC APPROXIMATION

If the crystal is subjected to a deformation, which is described by a strain tensor u_{ik} (for deformations which are not too strong)

$$u_{ik} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \quad (1)$$

u_i — components of the displacement vector, x_i — components of the position vector; then the frequencies of thermal phonons ω in the quasiharmonic approximation are the strain tensor functions

$$\omega(\vec{q}, s) = \omega_0(\vec{q}, s) [1 - \gamma_{ik}(\vec{q}, s) u_{ik}] \quad (2)$$

(summation is understood over the repeated indices), \vec{q} is the wave vector of the phonon, s indicates the polarisation, ω_0 is the frequency in the undeformed crystal, and $\gamma_{ik}(\vec{q}, s)$ is the generalized Grüneisen constant. In accordance with Woodruff and Ehrenreich [5], the phonon gas will be described by the distribution function $f(\vec{q}, s, \vec{r}, t)$, which measures the phonon density with the wave vector \vec{q} and with polarisation s at the position \vec{r} and at the time t . This description is valid as long as the phonon packet is localized within the displacement wave length and its frequency follows the deformation adiabatically.

For most phonons this holds if

$$\Omega < \frac{kT}{\hbar} \quad (3)$$

where k is the Boltzmann constant. The distribution function will be determined from the solution of the Boltzmann transport equation.

From the theory of elasticity it is known (see for instance [9]), that if σ_{ij} is the stress tensor then the motion equation is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad (4)$$

where ρ is the density. The stress tensor is determined by an adiabatic change of the internal energy of the crystal at an infinitesimal deformation

$$\sigma_{ij} = \left(\frac{\partial E}{\partial u_{ij}} \right) S = \text{const.} \quad (5)$$

where E is the internal energy and S the entropy of the unit volume. The internal energy of the system consists of the elastic energy $u_{el}(u_{ij})$ and of contributions of individual oscillators [10]

$$E = u_{el}(u_{ij}) + \frac{1}{2} \sum_{q,s} \hbar \omega(\vec{q}, s) + \sum_{q,s} \hbar \omega(\vec{q}, s) f(\vec{q}, s, \vec{r}, t) \quad (6)$$

The energy could have even other terms (anharmonic ones), but in the first approximation they are not functions of deformations and therefore do not contribute to the stress tensor. $\omega(\vec{q}, s)$ are eigen-frequencies of the deformed system and their dependence on the deformation is given by Eq. (2). The lowest approximation for u_{el} is given by the Hook law, the elastic energy is a quadratic function of the strain tensor

$$u_{el}(u_{ij}) = u_{el}(0) + \frac{1}{2} C_{ijkl} u_{ij} u_{kl} \quad (7)$$

where C_{ijkl} are elastic constants. The entropy of the nonequilibrium phonon gas is

$$S = k \sum_{q,s} [(f+1) \ln(f+1) - f \ln f] \quad (8)$$

hence the derivation (5) has to be done at constant f . Now we can write the motion equation (4) in a more explicit form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} - \frac{\partial}{\partial x_j} \sum_{q,s} \hbar \omega(\vec{q}, s) \gamma_{ij}(\vec{q}, s) f(\vec{q}, s, \vec{r}, t). \quad (9)$$

From this equation we see that the undamped acoustic wave may propagate through a crystal only at zero temperature and then it is described by the elastic constants C_{ijkl} .

Equations (9) together with the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} \frac{\partial \omega}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial \omega}{\partial x_i} = - \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \quad (10)$$

form a system of equations for our task. However, to solve equation (10) we must know the explicit form of the collision term. We express the collision term by means of the relaxation times for the normal phonon-phonon collisions $\tau_N(\vec{q}, s)$ (the N -processes), which conserve the phonon wave vector at the collision and relaxation times for the other processes $\tau_A(\vec{q}, s)$, which do not conserve the phonon wave vector. Both these processes relax to a local equilibrium distribution function, which is characterized by the local temperature $T(\vec{r}, t)$ and in the case of the N -processes this distribution is shifted in the phonon wave vector space away from the origin. This expression of the collision term which is based on the work of Callaway [8] was used also by Woodruff and Ehrenreich [5].

The distribution function towards which the N -processes relax can therefore be expressed as

$$f_0^{(N)} = \{ \exp [(\hbar\omega - \vec{\lambda} \cdot \vec{q}) / kT] - 1 \}^{-1} \quad (11)$$

and the distribution function for the other processes as

$$f_0^{(A)} = \{ \exp [\hbar\omega / kT] - 1 \}^{-1} \quad (12)$$

The parameter $\vec{\lambda}$, which in our case is also a function of \vec{r}, t determines the shifted distribution function discussed by Callaway [8] and Klemens [18].

The quantities T and $\vec{\lambda}$ are determined by the following conditions: By the effect of mutual collisions of the thermal phonons in the first approximation their unperturbed thermal energy is not changed and therefore we have

$$\sum_{\vec{q}, s} \hbar\omega(\vec{q}, s) \left(\frac{\partial f(\vec{q}, s, \vec{r}, t)}{\partial t} \right)_{\text{coll}} = 0. \quad (13)$$

For the normal processes, the conservation law of the wave vector holds, from which we get

$$\sum_{\vec{q}, s} \vec{q} \left(\frac{\partial f}{\partial t} \right)_{\text{coll}, N} = 0. \quad (14)$$

Introducing the local equilibrium phonon distribution we use the temperature $T(\vec{r}, t)$, which determines the temperature of the system in a small region around the point \vec{r} and at a small time interval near the time t . The minimum

magnitudes of this region and of this interval are given by the mean free path and by the mean life time of the phonons. However, to understand \vec{r} as point, or t as time moment, this region must be smaller than the sound wave length, or this time interval shorter than the sound period, that is, $\Omega r < 1$ must hold (the phonon mean free path equals $v\tau$, where v is the mean sound velocity). It is obvious that if $\Omega r > 1$ the local equilibrium states do not exist in the crystal and thus T approaches T_0 , as Ωr becomes greater than unity. This incompleteness of the conditions (13) and (14) does not cause real difficulties, because, as we shall see below, the expressions containing $(T - T_0)$ and $\vec{\lambda}$ disappear from our results as Ωr becomes greater than unity. T_0 is the temperature of the undeformed crystal.

To the first order in \vec{u} , $(T - T_0)$ and $\vec{\lambda}$ we have

$$f_0^{(N)} = f_0 + \frac{T_0 C(\vec{q}, s)}{\hbar\omega_0} \left(\gamma_{ij} u_j + \frac{T - T_0}{T_0} + \frac{\lambda \cdot \vec{q}}{\hbar\omega_0} \right) \quad (15)$$

$$f_0^{(A)} = f_0 + \frac{T_0 C(\vec{q}, s)}{\hbar\omega_0} \left(\gamma_{ij} u_j + \frac{T - T_0}{T_0} \right) \quad (16)$$

where f_0 is the phonon distribution function in the thermal equilibrium and

$$C(\vec{q}, s) = - \frac{\hbar\omega_0^2}{T_0} \frac{\partial f_0}{\partial \omega_0}$$

is the specific heat of the unit volume due to the mode (\vec{q}, s) in the undeformed crystal, therefore $\sum_{\vec{q}, s} C(\vec{q}, s) = C$ is the specific heat at the constant volume per unit volume. Further we will assume that the sound wave propagates in the direction of the z axis, which we identify with one of the principal crystal directions, that is with the direction in which the pure longitudinal or transverse wave can propagate, then the displacement has the form $u_i = e_i u$, where e_i are polarisation vector components and

$$u = u_0 \exp [i(\Omega t - \sigma z)] \quad (17)$$

where u_0 is the amplitude, Ω is the frequency and σ the sound wave vector. Thus the wave equation (9) and the transport equation (10) can be rewritten in the following form

$$\frac{\partial^2 u}{\partial z^2} = \epsilon \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z} \sum_{\vec{q}, s} f(\vec{q}, s, z, t) \hbar\omega_0(\vec{q}, s) \gamma(\vec{q}, s) \quad (18)$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial z} v_z - \frac{\partial f}{\partial q_z} \frac{\partial \omega}{\partial z} = - \left(\frac{\partial f}{\partial t} \right)_{\text{coll}} \quad (19)$$

where $\varepsilon = C_{12}z_2^2 e_{12}$, $\gamma = \gamma z_1^2 e_1$

$$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}} = \frac{f - f_0}{\tau} - \frac{T_0 C(\vec{q}, s)}{\hbar \omega_0 \tau} \left(\gamma \frac{\partial u}{\partial z} + \frac{\tau}{\tau_N} \frac{\lambda q_z}{\hbar \omega_0} + \frac{T - T_0}{T_0} \right) \quad (20)$$

$v_z(\vec{q}, s)$ is the group velocity of the thermal phonons in the direction of the sound propagation and

$$[\tau(\vec{q}, s)]^{-1} = [\tau_R(\vec{q}, s)]^{-1} + [\tau_N(\vec{q}, s)]^{-1}. \quad (21)$$

Equations (18), (19) together with the conditions (13), (14) and with the relation (20) make the system of equations of our problem, which we shall solve. If we retain in the Boltzmann equation (19) only the terms of the first order and we seek the solution in the form $f - f_0 \sim \exp[i(\Omega t - \sigma z)]$, then its solution is the function

$$f - f_0 = \frac{T_0 C(\vec{q}, s)}{\hbar \omega_0} \left(-\tau v_z \sigma^2 u - i \gamma \sigma u + \frac{T - T_0}{T_0} + \frac{\tau}{\tau_N} \frac{\lambda q_z}{\hbar \omega_0} \right) [1 + i \tau (\Omega - \sigma v_z)]^{-1} \quad (22)$$

The absorption coefficient of the sound amplitude can now be determined from equation (15) as an imaginary part of the wave vector (assuming that $\sigma \gg \alpha$)

$$\alpha = \frac{T_0 \sigma^2}{2\varepsilon} \sum_{\vec{q}, s} \gamma^2 C(\vec{q}, s) \operatorname{Re} Z \quad (23)$$

and the effective elastic constant is

$$\varepsilon' = \frac{\Omega^2 \rho}{\sigma^2} = \varepsilon - T_0 \sum_{\vec{q}, s} \gamma^2 C(\vec{q}, s) - T_0 \sum_{\vec{q}, s} \gamma^2 C(\vec{q}, s) \operatorname{Im} Z, \quad (24)$$

where

$$Z = \frac{-(T - T_0) - \frac{\tau}{\tau_N} \frac{\lambda q_z}{\hbar \omega_0} + \Omega \tau}{T_0 v_z \sigma} \frac{1}{1 + i \tau (\Omega - \sigma v_z)}.$$

From the conditions (13) and (14) we get

$$\sum_{\vec{q}, s} \frac{C(\vec{q}, s)}{1 + i \tau (\Omega - \sigma v_z)} \left[\Omega \gamma \sigma u + i (\Omega - \sigma v_z) \left(\frac{T - T_0}{T_0} + \frac{\tau}{\tau_N} \frac{\lambda q_z}{\hbar \omega_0} \right) \right] = 0 \quad (25)$$

$$\sum_{\vec{q}, s} \frac{\vec{q} C(\vec{q}, s)}{\hbar \omega_0 \tau_N} [1 + i \tau (\Omega - \sigma v_z)]^{-1} \left\{ -\Omega \tau \gamma \sigma u + \left[\frac{\tau}{\tau_N} - 1 - i \tau (\Omega - \sigma v_z) \right] \times \right.$$

$$\times \left. \frac{\lambda q_z}{\hbar \omega_0} - i \tau (\Omega - \sigma v_z) \frac{T - T_0}{T_0} \right\} = 0. \quad (26)$$

Our formula (24) for the elastic constants completes the results of Woodruff and Ehrenreich [5], which we have obtained as the relations (23), (25) and (26).

SPECIAL CASES

In this section, we shall determine on the basis of the relations (23–26) α and ε' in the following cases: a) $\Omega \tau_R \ll 1$; b) $\Omega \tau_R \gg 1 \gg \Omega \tau_N$; c) $\Omega \tau > 1$. However, not even these limitations are sufficient to solve the problem. Therefore we shall accept some simplifications. We shall assume that γ and τ are even functions of \vec{q} ; then for the expressions of the following type we have for $m + n$ odd

$$\sum_{\vec{q}, s} F(\gamma, \tau, \omega_0) v_z^m q_z^n = 0 \quad (27)$$

where $F(\gamma, \tau, \omega_0)$ is some function of γ , τ and ω_0 .

a) $\Omega \tau_R \ll 1$.

$$\alpha = \frac{CT_0 \Omega \sigma}{2\varepsilon} \langle \tau \langle \gamma \rangle - \gamma \rangle^2 + \frac{\sigma^3 T_0 \langle \gamma \rangle^2}{2\varepsilon \Omega} K_{zz} \quad (28)$$

$$\varepsilon' = \varepsilon - CT_0 \langle \gamma^2 \rangle - \langle \gamma \rangle^2 + O[(\Omega \tau_R)^2]. \quad (29)$$

In these expressions the triangular brackets represent the following average

$$\langle a \rangle = \sum_{\vec{q}, s} C(\vec{q}, s) a(\vec{q}, s) / C \quad (30)$$

and K_{zz} is the z - z component of the steady-state thermal conductivity

$$K_{zz} = \sum_{\vec{q}, s} v_z^2 C(\vec{q}, s) + \left[\sum_{\vec{q}, s} \frac{v_z q_z}{\omega_0} \frac{\tau}{\tau_N} C(\vec{q}, s) \right]^2 \left/ \sum_{\vec{q}, s} \frac{q_z^2 C(\vec{q}, s)}{\omega_0^2} \frac{1}{\tau_N} \left(1 - \frac{\tau}{\tau_N} \right) \right. \quad (31)$$

This expression is just the generalized Callaway [8] expression of the thermal conductivity. The tensor of the thermal conductivity \vec{K} is determined from the following expression

$$\vec{Q} = -\vec{K} \cdot \text{grad } T = i \vec{K} \cdot \vec{\sigma} (T - T_0) \quad (32)$$

where \vec{Q} is the heat current density vector

$$\bar{Q} = \hbar \sum_{q,s} \omega_0 \bar{v}(\bar{q}, s, \bar{r}) \quad (33)$$

where for f we must take the function (22) in the limit $\Omega\tau \rightarrow 0$ and $v_2\sigma\tau \ll 1$ (low temperature gradients). The absorption coefficient (28) differs from the results obtained by Woodruff and Ehrenreich [5] and Guyer [21], (apart from being more generally derived) in the first term, which is also called the Akhiezer attenuation [22, 23] if we substitute for ϵ its approximate expression $\epsilon = w^2 Q$, where w is the phase sound velocity in the direction of the z axis.

The expression (28) holds for the longitudinal as well as the transverse wave, where for the polarised wave in the direction \hat{e} and propagating in the z direction $\gamma = \gamma_2 \hat{e} \hat{e}$. In the Debye model γ can be expressed by means of elastic constants of the second and third order [11, 12]. From the works of Mason and Bateman [12] it follows that for the cubic crystals and the transverse wave (4) $\langle \gamma \rangle = 0$. From this it follows that

$$\alpha = \frac{CT_0 \Omega \sigma}{2\epsilon} \langle \gamma^2 \tau \rangle \quad (35)$$

$$\epsilon' = \epsilon - CT_0 \langle \gamma^2 \tau \rangle \quad (35)$$

while for the longitudinal wave the absorption and elastic constants are given by the whole expressions (28) and (29). The expression (29) in the adiabatic limit $\Omega\tau \rightarrow 0$ gives the adiabatic elastic constants. For the isothermal elastic constants ($T = T_0$, $\Omega\tau \rightarrow 0$) we have

$$\begin{aligned} \epsilon_{1s} &= \epsilon - CT_0 \langle \gamma^2 \tau \rangle \\ \epsilon_{3d} - \epsilon_{1s} &= CT_0 \langle \gamma \rangle^2. \end{aligned} \quad (36)$$

These are well-known relations from the theory of anharmonicity [13]. Consequently in the limit $\Omega\tau_R \ll 1$, the propagation of sound is described by the adiabatic elastic constant and by the absorption coefficient (28).

b) $\Omega\tau_R \gg 1 \gg \Omega\tau_N$ (the second sound region, [21] and references therein). In the next two cases b) and c) we shall accept the simplifying assumption about the energy spectrum of phonons. To explain the thermal conductivity [19] and absorption of sound [20] in InSb, the following model of the phonon spectrum turned out to be convenient

$$\omega_0(\bar{q}, s) = \begin{cases} v_s q & \text{for } q_s^{(1)} \leq q \leq q_s^{(2)} \\ 0 & \text{for other values of } q \end{cases} \quad (37)$$

where v_s is the average phonon velocity of the mode s with the wave vector from the interval $\langle q_s^{(1)}, q_s^{(2)} \rangle$. Now the index s does not indicate the branches only, but the parts of the branches too, where it is possible to approximate

the phonon spectrum by the linear relation (37). In this region we must distinguish between two cases, which are determined by the following quantity

$$p^2 = \frac{1}{3w^2} \left\langle \frac{1}{v^2} \right\rangle^{-1}. \quad (38)$$

If $p^2 = 1$, that is, the acoustic wave velocity is equal to the second sound velocity, this wave is strongly absorbed and we cannot directly apply expressions (23) and (24) which were derived from (18) under the assumption of low absorption. Therefore in this case we must start directly from equation (18), then,

$$\alpha = \sigma \frac{\epsilon_{1s}}{X} \left[1 + X^2/\epsilon_{1s}^2 \right]^{\frac{1}{2}} - 1 \quad (39)$$

$$\frac{\epsilon'}{\epsilon_{1s}} = \frac{2\alpha}{\sigma} \left(\frac{\epsilon_{1s}}{X} + \frac{X}{\epsilon_{1s}} \right) \quad (40)$$

where we have used the following notation

$$X = CT_0 \left(\frac{\langle \gamma \rangle^2}{\Delta} + \Omega \langle \tau_N \gamma^2 \rangle \right) \quad (41)$$

and

$$\Delta = \left\langle \frac{1}{v^2} \right\rangle^{-1} \left[\left\langle \frac{1}{v^2 \Omega \tau_R} \right\rangle + \left\langle \frac{\Omega \tau_N}{v^2} \right\rangle + \left\langle \frac{1}{v^2} \right\rangle^2 \langle v^2 \Omega \tau_N \rangle - \frac{6}{5} \left\langle \frac{1}{v^2} \right\rangle \langle \Omega \tau_N \rangle \right] \quad (42)$$

and ϵ_{1s} is given by Eq. (37). For the interaction of the acoustic wave with only one type of phonons ($s = 1$) we have $p^2 = v^2/3w^2$ and $\Delta = \frac{4}{5} \langle \Omega \tau_N \rangle + \langle \Omega \tau_R \rangle^{-1}$. This Δ ($s = 1$) differs from the Δ used by Guyer [21] only by the numerical coefficient at $\Omega \tau_N$. If $\langle \gamma \rangle = 0$, the sound absorption will be small and for α and ϵ' we get

$$\alpha = \frac{CT_0 \Omega \sigma}{2\epsilon} \langle \gamma^2 \tau_N \rangle; \quad \epsilon' = \epsilon_{1s}; \quad p = 1; \quad \langle \gamma \rangle = 0. \quad (43)$$

For $\langle \gamma \rangle \neq 0$, X given by Eq. (41) will be determined by the first term, which is now much greater than the second and if it is possible to put $X \ll \epsilon_{1s}$, we get

$$\alpha = \frac{T_0 \Omega \sigma \langle \gamma \rangle^2}{2\epsilon_{1s} \Delta}, \quad \epsilon' = \epsilon_{1s}, \quad p = 1. \quad (44)$$

This absorption coefficient corresponds with the one derived by Guyer [21]. If $p \neq 1$,

$$\alpha = \frac{CT_0\Omega\sigma}{2\epsilon} \langle \tau_N \langle \gamma \rangle - \gamma \rangle^2 + \frac{CT_0\sigma \langle \gamma \rangle^2}{2\epsilon} \times \\ \times \frac{p^2}{(1-p^2)^2} \Delta - \frac{2}{5} \frac{CT_0\Omega\sigma}{\epsilon} \langle \gamma \rangle^2 \langle \tau_N \rangle \quad (45)$$

$$\epsilon' = \epsilon - CT_0 \left(\langle \gamma^2 \rangle - \frac{\langle \gamma \rangle^2}{1-p^2} \right). \quad (46)$$

The expression (45), apart from containing the Akhiezer attenuation (the first term), differs from the result of Gulyaev [21] also in the last term. From Eqs. (43) and (44) we see that the acoustic wave propagating through the crystal with the velocity equal to that of the second sound is characterized by isothermal elastic constants. But for the waves with $p \neq 1$ the elastic constants are given by the expression (46).

$\Omega\tau > 1$. In this region we also use the approximation (37). By solving equations (25) and (26) we can see that the terms in the expressions (23) and (24) containing $(T - T_0)$ and λ are considerably smaller than $\Omega\tau$. Consequently the above mentioned incompleteness of the conditions (25) and (26) is not essential. In this region we obtain such values of α and ϵ' , which we could obtain if $T = T_0$ and $\lambda = 0$

$$\alpha = \frac{T_0\Omega}{8\pi^2\epsilon} \sum_s \int_{q_0^{(s)}}^{q_0^{(2s)}} dq q^2 \frac{\gamma^2}{v_s} C(\vec{q}, s) [\text{arctg} \Omega\tau(\beta_s - 1) + \text{arctg} \Omega\tau(\beta_s + 1)], \beta_s = \frac{v_s}{w} \quad (44)$$

$$\epsilon' = \epsilon - CT_0 \langle \gamma^2 \rangle + \frac{T_0}{8\pi^2} \sum_s \int_{q_0^{(s)}}^{q_0^{(2s)}} dq q^2 \frac{\gamma^2}{\beta_s} \ln \frac{1 + \Omega^2\tau^2(1 + \beta_s)^2}{1 + \Omega^2\tau^2(1 - \beta_s)^2}. \quad (45)$$

For the values of $s = 1$ and $\beta = 1$, the expression (44) is the same as the absorption coefficient of Woodruff and Ehrenreich [5]. The expression (44) corresponds also with the results obtained by quantum mechanical methods [14]. Next, we shall study the results (44) and (45) in a simple Debye model, that is $s = 1$, $v_s = v$, $q_0^{(1)} = 0$, $q_0^{(2)} = q_D$, $\gamma = \text{const.}$ and we shall distinguish the transverse (t) and the longitudinal (l) waves with $\beta_t = v/w_t$, $\beta_l = v/w_l$. As $\beta_l < 1$ and $\beta_t > 1$ in the limit $\Omega\tau \gg 1$ we obtain

$$\alpha_t = \frac{\pi}{4} \frac{CT_0\Omega}{w_l^3\varrho} \frac{\gamma_t^2}{\beta_t} \quad (46)$$

$$\alpha_l = \frac{1}{2} \frac{CT_0\gamma_l^2}{w_l^3\varrho} \left\langle \frac{1}{\tau} \right\rangle \frac{1}{1 - \beta_l^2} \quad (47)$$

where we have taken $\epsilon = w^2\varrho$. For $\beta = 1$

$$\alpha = \frac{\pi}{8} \frac{CT_0\Omega\gamma^2}{w^3\varrho}. \quad (48)$$

The absorption coefficient for the transverse wave (46) in this limit is independent on τ , it contains the frequency in the first power and the temperature as T_0C , which is the result corresponding to the theory of Landau and Rumer [1]. The absorption coefficient for the longitudinal wave (47) does not depend on the frequency and its temperature dependence is given by T_0C/τ , which corresponds to a higher temperature dependence than T_0C , since with decreasing temperature, the value of τ increases. It is obvious that the general formula (44) can contain the terms of all three types (46—48). Thus we come to the conclusion that the absorption coefficient for the wave of the highest frequency ($\beta_s < 1$ for all s) has a higher temperature dependence and a lower frequency dependence than the absorption coefficient for the slowest wave ($\beta_s > 1$ for all s). This is in agreement with the experiment [15].

For the elastic coefficients we obtain

$$\epsilon' = \epsilon - \gamma^2 T_0 C \left[1 - \frac{1}{4\beta} \ln \frac{(1 + \beta)^2}{(1 - \beta)^2} \right], \text{ for } \beta \neq 1 \quad (49)$$

$$\epsilon' = \epsilon - \gamma^2 T_0 C + \gamma^2 T_0 C \langle \ln 2\Omega\tau \rangle / 2, \text{ for } \beta = 1. \quad (50)$$

In the first case, where $\beta \neq 1$, we see from the expression (49) that $\epsilon' - \epsilon$ depends on the temperature as T_0C while in the second case when $\beta = 1$ this dependence on the temperature is determined also through the relaxation time τ . However, in any case the last term decreases with decreasing temperature. Thus for the U -processes $\tau \sim \exp \Theta/wT_0$, the last term depends on the temperature as a specific heat C .

CONCLUSION

Most results obtained in this paper are not new. However, the characteristics of the sound propagation in the crystal, that is, the absorption coefficient and the elastic constants, were obtained from an unified point of view. Our approach made it possible to determine these characteristics from the solution of the Boltzmann transport equation (10) and from the wave equation in quasi-harmonic approximation (9) which were complemented by the conservation of energy law (13) and conservation of momentum law (14) at normal collisions. The absorption coefficients in all three limit cases were derived with a greater generality than it has been done so far [5, 21] and were complemented by the

Akhiezer attenuation. The main new contribution of this paper is the determination of the elastic constants in the following cases $\Omega\tau_R \gg 1 \gg \Omega\tau_V$ and $\Omega\tau > 1$ from the general formula derived here.*)

REFERENCES

- [1] Landau L., Rumer G., Phys. Z. Sowjet 11 (1937), 18.
- [2] Ziman J. M., *Electrons and Phonons*. Oxford University Press, Oxford 1962
- [3] Simons S., Proc. Phys. Soc. (GB) 82 (1963), 401; *ibid* 83 (1964), 749.
- [4] Maris H. J., Phil. Mag. 9 (1964), 901.
- [5] Woodruff T. O., Ehrenreich H., Phys. Rev. 123 (1961), 1553.
- [6] Bömmel H., Dransfeld K., Phys. Rev. 117 (1960), 1245.
- [7] Akhiezer A., J. Phys. USSR 1 (1939), 277.
- [8] Callaway J., Phys. Rev. 113 (1959), 1046; *Proceedings of Int. Conf. on Semiconductor Physics*, Academia, Prague 1960, 627.
- [9] Ландау Л. Д., Лифшиц Е. М., *Теория жидкостей*. Изд. Наука, Москва 1965.
- [10] Leibfried G., in *Handbuch der Physik*. Edited by S. Flügge, Springer Verlag, Berlin 1955, Vol. VII, Part 2.
- [11] Brugger K., Phys. Rev. 137 (1965), A 1826.
- [12] Mason W. P., Bateman T. B., J. Acoust. Soc. Am. 36 (1964), 644; *ibid* 40 (1966), 852.
- [13] Leibfried G., Ludwig W., in *Solid State Physics*. Edited by F. Seitz and D. Turnbull, Academic Press Inc., New York --- London 1961, Vol. 12.
- [14] Miller P. B., Phys. Rev. 137 (1965), A 1937.
Гурьян Л. Э., Шкродский Б. И., ФТТ 9 (1967), 526.
- [15] Nava R., Azrt R., Ciccarelo I., Dransfeld K., Phys. Rev. 134 (1964) A 58
Pomerantz M., Phys. Rev. 139 (1965), A 501.
- [16] Götzte W., Michel K. H., Phys. Rev. 156 (1967), 963.
- [17] Peierls R. E., *Quantum Theory of Solids*. Oxford University Press, Oxford 1955.
- [18] Klemens P. G., in *Handbuch der Physik*. Edited by S. Flügge, Springer Verlag, Berlin 1956, Vol. XIX.
- [19] Bhandari C. M., Verma G. S., Phys. Rev. 140 (1965), A 2101.
- [20] Hrivnák L., Kovár J., phys. stat. sol. 23 (1967), 189.
- [21] Guyer R. A., Phys. Rev. 148 (1966), 789.
- [22] Klemens P. G., in *Physical Acoustics*. Edited by W. P. Mason, Academic Press, New York 1965, Vol. III, Part B, Chapt. 5.
- [23] Mason W. P., *ibid* Chapt. 6.

Received March 22^a, 1968

*Fyzikálny ústav SAV,
Bratislava*

*) The author would like to express his thanks to Dr. L. Hrivnák for valuable discussions.