**acta physica slovaca vol. 49 No. 4, 491 – 500** **August 1999**

# NONLOCALITY OF THE EINSTEIN-PODOLSKY-ROSEN STATE IN THE PHASE SPACE*

**K. Banaszek**[a1], **K. Wódkiewicz**[ab2]
*(a) Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, PL-00-681 Warszawa, Poland*
*(b) Abteilung für Quantenphysik, Universität Ulm, D-89081 Ulm, Germany*

Received 19 April 1999, accepted 10 May 1999

We discuss violation of Bell inequalities by the regularized Einstein-Podolsky-Rosen (EPR) state, which can be produced in a quantum optical parametric down-conversion process. We propose an experimental photodetection scheme to probe nonlocal quantum correlations exhibited by this state. Furthermore, we show that the correlation functions measured in two versions of the experiment are given directly by the Wigner function and the $Q$ function of the EPR state. Thus, the measurement of these two quasidistribution functions yields a novel scheme for testing quantum nonlocality.

PACS: 03.65.Bz, 42.50.Dv

## 1 Introduction

The work of Einstein, Podolsky, and Rosen (EPR) has started a lasting debate about the completeness and the meaning of local realities in quantum mechanics. In order to make their point, EPR used the following wave function for a system composed of two particles [1]:

$$|\mathrm{EPR}\rangle = \int \frac{dp}{2\pi}\, |p, -p\rangle = \int dq\, |q, q\rangle. \tag{1}$$

Compared to the original paper of EPR, we have put here spatial separation between the particles equal $q_0 = 0$. The above wave function describes an entangled state of two particles $a$ and $b$, with the following properties:

$$(\hat{q}_a - \hat{q}_b)|\mathrm{EPR}\rangle = 0, \quad (\hat{p}_a + \hat{p}_b)|\mathrm{EPR}\rangle = 0. \tag{2}$$

However, further studies of quantum nonlocality and entanglement, especially those providing quantitative tests of compatibility of quantum mechanics with local theories in the form of

*Presented at the 6th Central-European Workshop on Quantum Optics, Château Chudobín, Czech Republic, April 30 - May 3, 1999.

[1]E-mail address: Konrad.Banaszek@fuw.edu.pl
[2]E-mail address: Krzysztof.Wodkiewicz@fuw.edu.pl

 

Bell inequalities, used mainly spin-1/2 particles instead of systems with continuous degrees of freedom.

Quantum correlations for position-momentum variables associated with the EPR state (1) can be analyzed in phase space using the Wigner function or the positive $Q$ function. Using the Wigner function approach, Bell [2] has argued that the EPR wave function (1) will not exhibit nonlocal effects, because its Wigner function is positive everywhere, and as such will allow for a hidden variable description of the correlations. This statement is correct as long as the measured observables have a straightforward classical interpretation in the phase space representation. However, the situation can change dramatically, if we take into account quantum observables for which the phase space cannot serve as a local theory model. In a recent publication [3] we have demonstrated that the Wigner function of the EPR state, though positive definite, provides a direct evidence of the nonlocal character of this state. The demonstration was based on the fact that the Wigner function can be interpreted as a correlation function for the joint measurement of the parity operator.

In this presentation we review the EPR state in the Wigner representation and extend our approach to the positive $Q$ function, which also can be regarded as a correlation function for the joint measurement of certain dichotomic observables. The analysis will be performed for an optical realization of the EPR state as the entangled two-mode state of light generated in the spontaneous parametric process. In this quantum optical context, analysis of the $Q$ function is of particular interest, as the experimental demonstration of nonlocal correlations exhibited by the $Q$ function poses less stringent requirements on single photon detectors used in the setup.

This paper is organized as follows. First, in Section 2, we briefly review the quantum optical version of the EPR state. In Section 3 we discuss the Wigner and the $Q$ functions of this state, and study the limiting case in which the original EPR state is recovered. In Section 4 we present the quantum optical scheme which can be used to demonstrate nonlocality of the EPR state in the phase space. Finally, Section 5 concludes this presentation.

## 2 EPR state in quantum optics

Before we show that the phase space representation of the EPR state fully exhibits the nonlocal character of this entangled state, we shall give a brief description of an optical realization of this state, in terms of the two-mode correlated light generated in the spontaneous parametric down-conversion process.

With a clear application to quantum optics, it has been shown that a state produced in a process of nondegenerate optical parametric amplification (NOPA) [4], is the optical analog of the EPR state in the limit of strong squeezing. The NOPA process is a nonlinear interaction of two quantized modes (denoted by $a$ and $b$) in a nonlinear medium with a strong classical pump field. The strength of the interaction can be characterized with the parameter $\chi$, which involves the second-order susceptibility and the pump field amplitude. The interaction Hamiltonian of the system is:

$$H = i\chi(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}). \tag{3}$$

If the initial state of the system consists of two vacuum modes the NOPA generates:

$$|\mathrm{NOPA}\rangle = e^{r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})}|0, 0\rangle, \tag{4}$$

where $r = \chi t$ is a dimensionless parameter characterizing the interaction time. Simple algebra, based on the following disentanglement of the evolution operator:

$$e^{r(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})} = e^{\tanh r\, \hat{a}^\dagger \hat{b}^\dagger} \left(\frac{1}{\cosh r}\right)^{\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1} e^{-\tanh r\, \hat{a}\hat{b}}, \tag{5}$$

shows that the generated state has a diagonal decomposition in terms of the number states of the two modes:

$$|\mathrm{NOPA}\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty} (\tanh r)^n |n, n\rangle. \tag{6}$$

In order to see the relation between this state and the EPR state (1), we rewrite the state (6) in the following form:

$$|\mathrm{NOPA}\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty} (\tanh r)^n \int dq \int dq'\, |q, q'\rangle\langle q, q'|n, n\rangle. \tag{7}$$

Using the fact that the scalar products can be expressed in terms of the Hermite polynomials (in dimensionless units): $\langle q|n\rangle = (2^n n! \sqrt{\pi})^{-1/2} H_n(q) \exp(-q^2/2)$, and the following summation formula (valid for arbitrary $\lambda \le 1$):

$$\sum_{n=0}^{\infty} \lambda^n \langle q|n\rangle\, \langle n|q'\rangle \;=\; \frac{1}{\sqrt{\pi(1-\lambda^2)}}\; \exp\left(-\frac{q^2 + q'^2 - 2\lambda q q'}{2(1-\lambda^2)}\right), \tag{8}$$

we obtain:

$$|\mathrm{NOPA}\rangle = \frac{1}{\sqrt{\pi}} \int dq \int dq' \exp\left(-\frac{q^2 + q'^2 - 2qq' \tanh r}{2(1-\tanh^2 r)}\right) |q, q'\rangle. \tag{9}$$

This formula is a regularized version of the EPR state (1), with a gaussian smoothing profile of the plane waves. Now it becomes clear, that in the limit of $r \to \infty$ i.e., for a very long interaction time, this smoothing function becomes a sharp function: $\delta(q - q')$. This follows from the fact that for $\lambda = 1$, the sum (8) reduces to the completeness relation for the oscillator wave functions. In this limit the state (9) becomes exactly the EPR state (1), and as result we obtain, that in terms of photons, the EPR state is:

$$|\mathrm{EPR}\rangle \sim \lim_{r\to\infty} |\mathrm{NOPA}\rangle \sim |0, 0\rangle + |1, 1\rangle + |2, 2\rangle + \dots . \tag{10}$$

The NOPA state has been generated experimentally [5] and applied to discuss the implications of the positivity of the corresponding Wigner function on the Bell inequality [6].

## 3 EPR state in phase space

In this section, we discuss the Wigner and the $Q$ functions of the state produced in the nondegenerate spontaneous parametric down-conversion process. We shall pay particular attention to the limit of strong interaction for $r \to \infty$, where the original, singular EPR state is recovered.

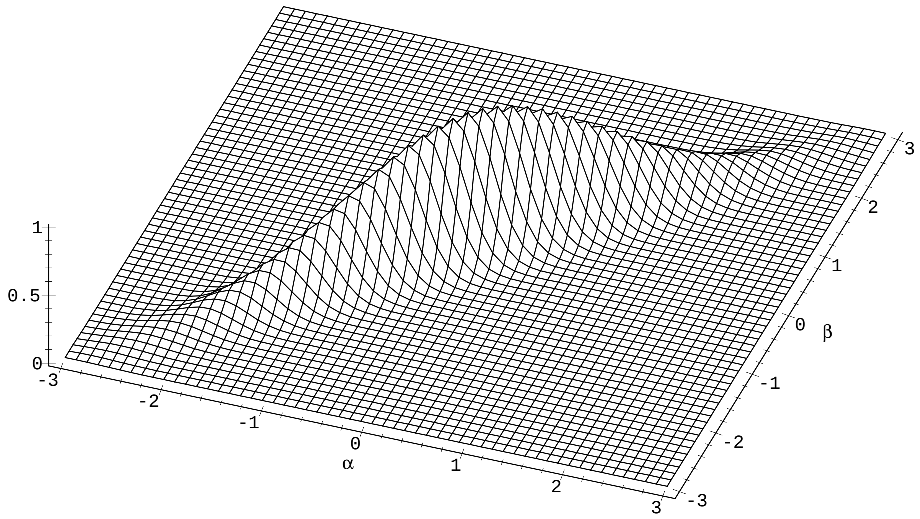


Fig. 1. Plot of the Wigner function of the regularized EPR state with $r = 1$ for real values of $\alpha$ and $\beta$.

### 3.1 The Wigner function in the coherent state representation

The two mode Wigner function of the NOPA state (9) can be calculated directly from the definition:

$$W(\alpha;\beta) = \int \frac{d^2\alpha'}{\pi^2} \int \frac{d^2\beta'}{\pi^2} \exp(\alpha\alpha'^* - \alpha^*\alpha' + \beta\beta'^* - \beta^*\beta')\langle \hat{D}_a(\alpha')\hat{D}_b(\beta')\rangle \,, \quad (11)$$

where $\hat{D}_a(\alpha)$ and $\hat{D}_b(\beta)$ are the Glauber's displacement operators for modes $a$ and $b$. Simple calculation shows that the Wigner function of the state (9) has the form:

$$W(\alpha;\beta) = \frac{4}{\pi^2}\exp[-2\cosh 2r(|\alpha|^2 + |\beta|^2) + 2\sinh 2r(\alpha\beta + \alpha^*\beta^*)]. \quad (12)$$

This positive everywhere Wigner function is plotted in Fig. 1 for real values of $\alpha$ and $\beta$ and for $r = 1$. The Wigner function of the original EPR state (1) is obtained in the limit $r \to \infty$:

$$W(\alpha;\beta) \sim \delta(\alpha_r - \beta_r)\delta(\alpha_i + \beta_i), \quad (13)$$

where $\alpha_r$ $(\alpha_i)$ and $\beta_r$ $(\beta_i)$ are the real (imaginary) parts of $\alpha$ and $\beta$. Note the singular form of this Wigner function, due to the singular character of the original EPR wave function (1).

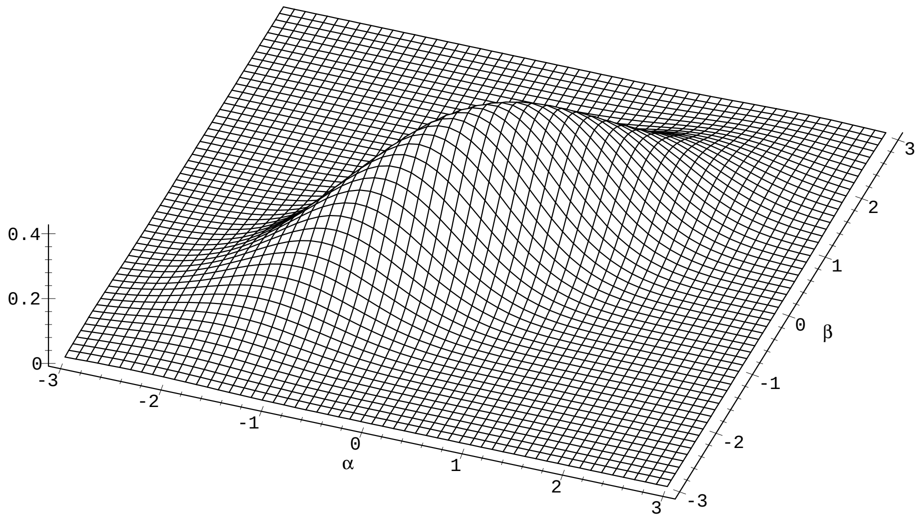


Fig. 2. Plot of the $Q$ function of the regularized EPR state for real values of $\alpha$ and $\beta$, for $r = 1$.

### 3.2 The $Q$ function in the coherent state representation

The two mode $Q$ function of the NOPA state can be calculated directly from its definition:

$$Q(\alpha;\beta) = \frac{1}{\pi^2}|\langle\alpha,\beta|\mathrm{NOPA}\rangle|^2. \tag{14}$$

It is easy to show that for the NOPA state (9) this function has the following form:

$$Q(\alpha;\beta) = \frac{1}{(\pi\cosh r)^2}\exp\left(-|\alpha|^2 - |\beta|^2 + \tanh r(\alpha^*\beta^* + \alpha\beta)\right), \tag{15}$$

with the following marginals:

$$\begin{aligned} Q(\alpha) &= \frac{1}{\pi(\cosh r)^2}\exp\left(-\frac{|\alpha|^2}{(\cosh r)^2}\right) \\ Q(\beta) &= \frac{1}{\pi(\cosh r)^2}\exp\left(-\frac{|\beta|^2}{(\cosh r)^2}\right). \end{aligned} \tag{16}$$

This two-mode $Q$ function is plotted in Fig. 2 for real values of $\alpha$ and $\beta$ and for $r = 1$. The $Q$ function of the original EPR state (1) obtained in the limit $r \to \infty$ is proportional to:

$$Q(\alpha;\beta) \sim \exp(-|\alpha|^2 - |\beta|^2 + \alpha^*\beta^* + \alpha\beta). \tag{17}$$

The last term in the exponent is responsible for the correlation of the entangled EPR state. Note that if we consider the $Q$ function given by Eq. (17) as a limit of Eq. (15), there is a vanishing prefactor with the magnitude proportional to $e^{-2r}$. The function (17) can be derived directly from the EPR wave function (1):

$$Q(\alpha,\beta) \sim \left| \int dq \langle q, q | \alpha, \beta \rangle \right|^2 . \tag{18}$$

This simple integration of the coherent state wave functions, in position representation, can be calculated and as a result we reproduce the result (17).

## 4 Testing quantum nonlocality in phase space

In this presentation we propose to probe the correlations of the NOPA state with a scheme involving two photon counting detectors for modes $a$ and $b$. The setup to demonstrate quantum nonlocality for the Wigner function and the $Q$ function is presented in Fig. 3. The photons from the NOPA source are measured by two photon counting detectors preceded by two beam splitters. These beam splitters, with the transmission coefficients very close to one, mix the NOPA photons with two highly excited coherent states. It has been shown elsewhere [7], that in the limit of the transmission tending to one, the effect of the beam splitters is equivalent to phase space shifts for the two modes by $\hat{D}_a(\alpha)$ and $\hat{D}_b(\beta)$, where $\alpha$ and $\beta$ are complex amplitudes of the reflected coherent fields. A single-mode version of such an experiment, involving measurements of the displaced photon number statistics for simple classical states of light, is reported in these proceedings in the context of quantum state measurement [8].

We shall show that if the detectors placed in the two arms of the proposed measurement scheme can resolve the number of absorbed photons, the phase space Wigner function can be determined directly. Moreover, it will become apparent that the Wigner function measured in this setup describes correlations between the parity of the number of photons registered by the two detectors. As the parity is a dichotomic $\pm 1$ variable, the Wigner function can be therefore directly inserted into appropriate Bell inequalities in order to test the nonlocal character of the NOPA state.

The most efficient single-photon detectors available currently are avalanche photodiodes operated in the Geiger regime. These detectors cannot resolve the number of simultaneously absorbed photons, and provide only a binary answer saying whether any photons have been registered or not [9]. We shall show that use of these detectors in our scheme leads in a natural way to the measurement of the two-mode $Q$ function, and that the binary answer of the detectors makes the $Q$ function a nonlocal correlation function which violates the corresponding Bell inequality.

### 4.1 Violation of local reality by the Wigner function

In order to relate the Wigner function to photocount correlations, we will consider the case when the detectors are capable of resolving the number of absorbed photons. Let us assign to

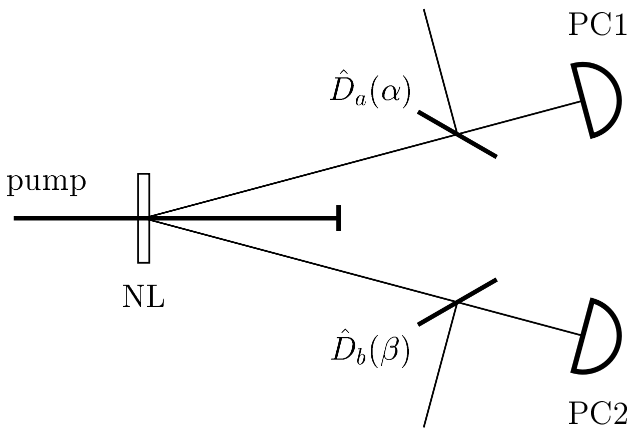


Fig. 3. The experimental scheme proposed to test nonlocality exhibited by the quantum optical realisation of the EPR state. The nonlinear crystal NL pumped by the coherent field generates two correlated beams. The high-transmission beam splitters with auxiliary coherent fields placed in the paths of these beams perform the displacement transformations $\hat{D}_a(\alpha)$ and $\hat{D}_b(\beta)$. The phase space displaced beams are monitored by the photon counting detectors PC1 and PC2.

each event $+1$ or $-1$, depending on whether an even or an odd number of photons has been registered. In this case the joint correlation for the photon number parities is:

$$E(\alpha;\beta) = \langle \mathrm{NOPA}|\hat{\Pi}_a(\alpha) \otimes \hat{\Pi}_b(\beta)|\mathrm{NOPA}\rangle, \tag{19}$$

where

$$\hat{\Pi}_a(\alpha) = \hat{D}_a(\alpha)(-1)^{\hat{n}_a}\hat{D}_a^\dagger(\alpha), \qquad \hat{\Pi}_b(\beta) = \hat{D}_b(\beta)(-1)^{\hat{n}_b}\hat{D}_b^\dagger(\beta), \tag{20}$$

are parity operators of the modes $a$ and $b$ characterized by the corresponding photon number operators $\hat{n}_a$ and $\hat{n}_b$. Equivalently, $E(\alpha;\beta)$ can be written as the overall parity operator displaced in the two-mode Hilbert space by the operation $\hat{D}_a^\dagger(\alpha) \otimes \hat{D}_b^\dagger(\beta)$. This observable is known in the phase space representation to provide, up to the normalization constant, the Wigner function of the quantum state [7]. On the other hand, we can establish a clear analogy of this scheme with joint spin-1/2 measurements. The counterpart of the polarizer orientation is now the coherent displacement, which can be set freely in each of the two spatially separated apparatuses. Furthermore, classification of the registered number of photons according to the parity provides a dichotomic $\pm 1$ outcome which is analogous to the spin direction. As a result all types of Bell's inequalities derived for spin systems can be used to test the nonlocality of the NOPA Wigner function. In Ref. [3] we have shown that the Wigner function (12) leads to a violation of the the Bell inequality. Here we will just quote the result. For local hidden variable theories the following combination:

$$\mathcal{B} = E(\alpha';\beta') + E(\alpha';\beta) + E(\alpha;\beta') - E(\alpha;\beta) \tag{21}$$

should satisfy the Bell inequality $-2 \le \mathcal{B} \le 2$. We have found that for a certain selection of coherent displacements the value of this combination in the limit $r \to \infty$ is: $\mathcal{B} = 1 + 3 \cdot 2^{-4/3} \approx 2.19$, which clearly violates the Bell inequality.

### 4.2 Violation of local reality by the $Q$ function

Based on the previous result we shall demonstrate how nonlocality of the EPR state can be revealed using the positive definite $Q$ quasidistribution function. The quantum optical experiment we shall propose does not require detectors that can resolve the number of photons triggering the output signal. We will be interested in events when *no photons* were registered. This measurement is described by the following observable:

$$\hat{O}_a = \lim_{\epsilon \to 0} (\epsilon)^{\hat{n}_a} = :e^{-\hat{n}_a}: = |0_a\rangle\langle 0_a|, \tag{22}$$

for the mode $a$, and a similar observable $\hat{O}_b$ for the mode $b$. As we discussed in the previous subsection, these observables can be shifted in phase space using auxiliary coherent fields and high-transmission beam splitters. As a result of this shifting we obtain:

$$\hat{O}_a(\alpha) = \hat{D}_a(\alpha)\hat{O}_a\hat{D}_a^\dagger(\alpha), \quad \hat{O}_b(\beta) = \hat{D}_b(\beta)\hat{O}_b\hat{D}_b^\dagger(\beta). \tag{23}$$

Obviously, these observables describe projections on a coherent state $|\alpha\rangle\langle\alpha|$ for the mode $a$ and $|\beta\rangle\langle\beta|$ for the mode $b$. Therefore, statistics of no-count events yields directly the $Q$ function of the measured field.

The quantum mechanical probability of no-count events in both the detectors is:

$$\begin{aligned} p_{ab}(\alpha;\beta) &= \langle \mathrm{NOPA}|\hat{O}_a(\alpha) \otimes \hat{O}_b(\beta)|\mathrm{NOPA}\rangle \\ &= \frac{1}{(\cosh r)^2} \exp\left(-|\alpha|^2 - |\beta|^2 + \tanh r(\alpha^*\beta^* + \alpha\beta)\right) \end{aligned} \tag{24}$$

where $\alpha$ and $\beta$ are the coherent displacements for the modes $a$ and $b$, respectively. The probabilities of no-count events on single detectors are:

$$\begin{aligned} p_a(\alpha) &= \langle \mathrm{NOPA}|\hat{Q}_a(\alpha) \otimes \hat{I}_b|\mathrm{NOPA}\rangle = \frac{1}{(\cosh r)^2} \exp\left(-\frac{|\alpha|^2}{(\cosh r)^2}\right), \\ p_b(\beta) &= \langle \mathrm{NOPA}|\hat{I}_a \otimes \hat{O}_b(\beta)|\mathrm{NOPA}\rangle = \frac{1}{(\cosh r)^2} \exp\left(-\frac{|\beta|^2}{(\cosh r)^2}\right). \end{aligned} \tag{25}$$

We recognize that these probabilities are equal up to a normalization factor the two-mode $Q$ function (15) and its two marginals for the NOPA state.

The correlation function $p_{ab}(\alpha,\beta)$ describes a binary 0 or 1 measurement on the modes $a$ and $b$ with adjustable parameters of the apparatuses represented by $\alpha$ and $\beta$. As a result of this, the Bell inequality derived for a measurement of local realities bounded by 0 and 1, can be applied to test the nonlocal character of the NOPA state (6), using the $Q$ function. We shall consider the the Clauser-Horne combination [10] for a selected set of four displacements:

$$\mathcal{CH} = p_{ab}(0;0) + p_{ab}(\alpha;0) + p_{ab}(0;\beta) - p_{ab}(\alpha;\beta) - p_a(0) - p_b(0), \tag{26}$$

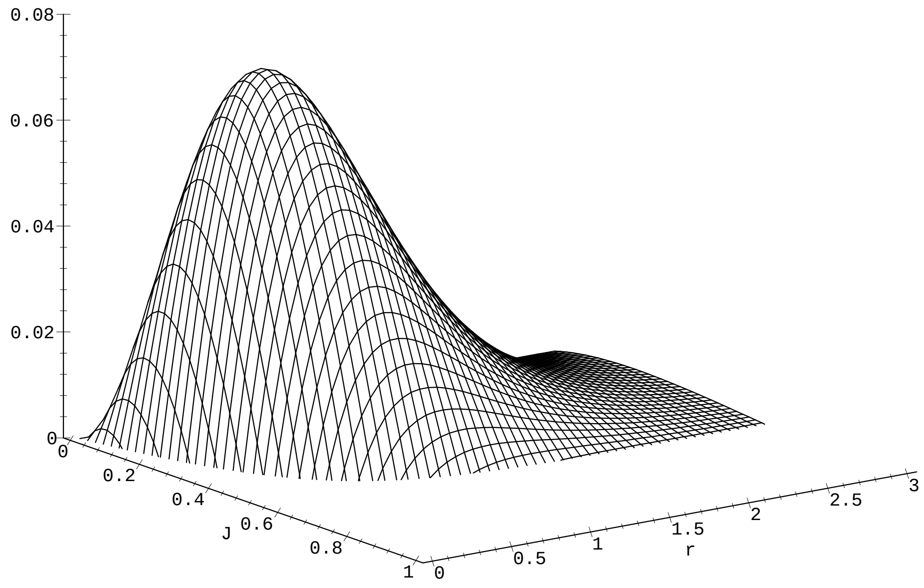


Fig. 4. Plot of the combination $\mathcal{CH}$ as a function of the displacement intensity $J$ and the interaction parameter $r$. Only values exceeding the upper bound imposed on $\mathcal{CH}$ by local theories are shown.

which for local theories satisfies the inequality $-1 \leq \mathcal{CH} \leq 0$. We will take the coherent displacements to be real with $\alpha = -\beta = \sqrt{J}$. For these values we obtain

$$\mathcal{CH} = \frac{1}{(\cosh r)^2} \left( 2e^{-J} - e^{-2J(1+\tanh r)} - 1 \right). \tag{27}$$

As depicted in Fig. 4, this result violates the upper bound imposed by local theories. In the limit of $r \to \infty$ and small values of the displacement, this function becomes:

$$\mathcal{CH} \approx 8Je^{-2r}, \tag{28}$$

i.e., it violates the Bell inequality for all values of $r$, but the violation becomes infinitesimally small. This results from the fact that the $Q$ function (17) in this limit is dumped by the prefactor $1/\cosh^2 r$, decreasing like $e^{-2r}$.

## 5 Conclusions

In this presentation we have shown that nonlocality of the EPR state can be revealed using its phase space representation of the form of the Wigner or $Q$ quasidistribution functions. This is possible owing to the observation that these quasidistribution functions describe nonlocal correlations that can be detected in a certain quantum optical scheme. Of course, this scheme is much more general. It can be applied to measure an arbitrary two-mode state of light, and the corresponding Wigner and $Q$ functions will always have the operational meaning of nonlocal correlation functions [11].

**Acknowledgments** This work has been partially supported by the Polish KBN grant No. 2 P03B 089 16. K.W. thanks the Alexander von Humboldt Foundation for generous support and Prof. W. P. Schleich for his hospitality in Ulm.

**References**

[1] A. Einstein, B. Podolsky, N. Rosen: *Phys. Rev.* **47** (1935) 777

[2] J. S. Bell: *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge, 1987) Chap. 21

[3] K. Banaszek, K. Wódkiewicz: *Phys. Rev. A* **58** (1998) 4345

[4] M. D. Reid, P. D. Drummond: *Phys. Rev. Lett.* **60** (1988) 2731

[5] Z. Y. Ou, S. F. Pereira, H. J. Kimble, K. C. Peng: *Phys. Rev. Lett.* **68** (1992) 3663

[6] Z. Y. Ou, S. F. Pereira, H. J. Kimble: *Appl. Phys. B* **55** (1992) 265

[7] K. Banaszek, K. Wódkiewicz: *Phys. Rev. Lett.* **76** (1996) 4344

[8] K. Banaszek, C. Radzewicz, K. Wódkiewicz, J. S. Krasiński: *quant-ph/9903027* (to appear in *Phys. Rev. A*); *acta physica slovaca* **49** (1999) 643, in this issue

[9] P. G. Kwiat, A. M. Steinberg, R. Y. Chiao, P. H. Eberhard, M. D. Petroff: *Phys. Rev. A* **48** (1993) 867

[10] J. F. Clauser, M. A. Horne: *Phys. Rev. D* **10** (1974) 526

[11] K. Banaszek, K. Wódkiewicz: *Phys. Rev. Lett.* **82** (1999) 2009