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MALUS' LAW AND QUANTUM INFORMATION*

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The information content of the most elementary quantum system is represented by one single proposition. Therefore such an elementary system can only give a definite result in one specific experimental arrangement. A change of experimental parameters then necessarily implies probabilistic measurement results in the new experimental arrangement. Assumption of the invariance of the information content of a system upon change of the representation of our knowledge of the system together with homogeneity of the experimental parametric axis leads to the Malus' law in quantum mechanics, the familiar sinusoidal relation between the probabilities and the laboratory parameters.

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1 Introduction

Quantum theory supplies a set of rules how physical conditions of an experimental arrangement determine the probabilities for different possible results of the experiment. The origin of these rules does not seem to be clear. But such is necessary if we want to be able to understand how we can know what physical conditions we prepared in an experiment from which in turn we can calculate the probabilities for different results.

Consider a beam of photons prepared in vertical polarization by a polarizing filter and analyzed by a Nicol prism as depicted in Fig. 1 schematically. Let $\theta \in [0, 180°[$ be the angle by which the Nicol prism has been rotated around the axis of the beam, starting from a standard position ($\theta = 0$) in which the prism's preferred axis is vertical. Each individual photon, when it encounters the Nicol prism, has exactly two options: to pass straight through the prism or to be deflected in a specific direction characteristic of the prism. Depending whether the upper or the lower detector plate was impinged we call the outcome "yes" and "no", respectively. Quantum theory predicts $p(\theta) = \cos^2\theta$ for the function between probability for a "yes" count and orientation of the Nicol prism. We ask: From what deeper foundation emerges this law, initially formulated by Malus [1] for light, in quantum mechanics.

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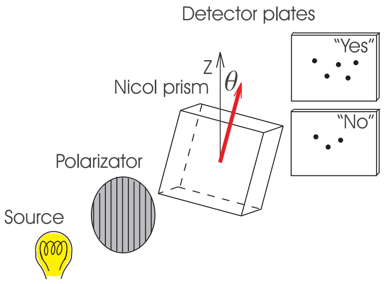

Fig. 1. Measurement of the polarization of the photon. Photon from the source passes through the vertical oriented polarizator, a Nicol prism oriented at the angle θ to the polarizator, and then it impinges one of the detector plates behind the Nicol prism. Depending whether the upper or the lower detector plate was impinged we call the outcome "yes" and "no", respectively. We ask: From what deeper foundation emerges the well-known function $p(\theta) = \cos^2\theta$ between the probability for a "yes" outcome and the angle of orientation of the Nicol prism?

2 Principle of Quantization of Information

Any physical system can be described by a set of propositions together with their truth values - true or false. Any propositions we might assign to a system are arrived only by observation and represent our knowledge, i.e., information, of an object gained through observation.

Consider encoding information into the polarization of photons. Let us consider our specific example of a photon with vertical polarization. The information content of the photon can then be expressed as a truth value of the proposition: "The polarization of the photon is vertical". This permits the prediction of individual outcomes with certainty only for the case when we actually perform a measurement in the vertical-horizontal polarization basis. Measurement for any other measurement basis must necessarily be probabilistic. We suggest that the reason for this is a principle of quantization of information [2]: *The most elementary quantum system represents the truth value of one proposition* only. This principle is then the reason for the irreducible randomness of an individual quantum event [2, 3] and for quantum entanglement [2, 4].

In our example the photon's vertical polarization represents the true value of exactly one proposition. Since this is the only information the photon's polarization carries, any proposition "The polarization of the photon is $+\theta$" for other orientations ($\theta \neq 0, 90°$) of the Nicol prism must necessarily contain an irreducible element of randomness. In particular, if we rotate the Nicol prism to 45° we have an extreme case with completely random measurement results. This kind of randomness must then be irreducible, that is, it cannot be reduced to "hidden" properties of the system. Otherwise the photon would not be elementary for polarization, it would carry enough information to assign truth values to more than one proposition.

3 Prediction of the Number of "Yes" and "No" Counts

Suppose an experimenter wants to predict how many photons result in each of the two possible outcomes knowing the probability for the outcomes to occur. In making her/his prediction she/he has only a limited number of photons to work with. Then, because of the statistical fluctuations associated to any finite number of experimental trials, the number n of occurrences of the outcome "yes" in N future repetitions of the experiment is not precisely predictable. More precisely, the experimenter's uncertainty (mean-square-deviation) in the value n is [5]

$$\sigma_N^2 = pqN. \tag{1}$$

We suggest to identify

$$U_N = 4\sigma_N^2 = 4pqN \tag{2}$$

as the experimenter's lack of information with respect to the number of occurrences of a certain outcome in N future experimental trials. This has the property of resulting in the minimal value of 0 if p equals zero and q equals one (or vice versa) and in the maximal value of N if $p = q = 1/2$. The property of proportionately to the number of performances guarantees that each individual performance of the experiment contributes the same amount of information gain, no matter how many times the experiment has already been performed. After each individual experimental trial the experimenter's information therefore increases by the amount

$$U = \frac{U_N}{N} = 4pq. \tag{3}$$

Clearly, the amount of information with respect to a single future experimental trial an experimenter possess *before* the experiment is performed will be specified by [3, 6]

$$I := 1 - U = (p - q)^2. \tag{4}$$

Given the probabilities for outcomes to occur the concept of probabilistic prediction of the future number of occurrences of outcomes is all an experimenter can predict in the open future which, in quantum mechanics, is fundamentally not precisely predictable. As specified by Eq.(4) the experimenter's prediction is more precise if p is close to 0 or 1, but much less when p is around 0.5.

The measure of information (4) is invariant under permutation of the set of possible outcomes. This property states that the amount of information is indifferent under certain changes of the experimental situation. In our example this results in an equal amount of information both for photons prepared initially in vertical polarization and in horizontal polarization. But these are different experimental situations. In order to remove this ambiguity we can additionally associate to each specific outcome its probability for occurrence or assign other distinct labels to possible outcomes, the particular scheme is of no further relevance. We use a quantity

$$i := p - q, \tag{5}$$

because it specifies also the amount of information by $I = i^2$. We call this quantity *information* with respect to a single specific measurement, because it is the whole information of a particular physical situation equivalent to the assignation of specific probabilities for each of the possible results.

4 The Total Information Content of a Quantum System

All the quantum state is meant to be is a representation of that catalog of our knowledge of the system that is necessary to arrive at the set of, in general probabilistic, predictions for all possible future observation of the system. In the view of Bohr's [7] remark that "... phenomena under different experimental conditions, must be termed complementary in the sense that each is well defined and that together they exhaust all definable knowledge about the object concerned" we describe a photon by a catalog of information ("information vector") $\vec{i} = (i_1, i_2)$ about mutually complementary propositions $\{\mathcal{P}_1, \mathcal{P}_2\}$. Such propositions are, for example, $\mathcal{P}_1$: "The polarization of the photon is vertical (horizontal)", and $\mathcal{P}_2$: "The polarization of the photon is $\pm 45°$". We now define the total information content I_{total} of a photon as the *sum* of the individual measures of information of the set of mutually complementary propositions[2]

$$I_{total} = I_1 + I_2 = i_1^2 + i_2^2 = 1. \tag{6}$$

An elementary limit of 1 to the total information content of a quantum system is implied by the principle of quantization of information. For the case of the vertically polarized photon $I_1 = 1$ and $I_2 = 0$.

We require that the total information content of a photon is *invariant* under the change of representation of the catalog of our knowledge of the system, that is, *independent* of the particular set of mutually complementary propositions considered. If we restrict ourselves to linear polarization any set of mutually complementary propositions may be written as $\mathcal{P}_1(\theta)$: "The polarization of the photon is $\pm\theta$" and $\mathcal{P}_2(\theta)$: "The polarization of the photon is $\pm(\theta + 45°)$". Here, the experimental situations of two corresponding experimental arrangements are labeled by a single experimental parameter which specifies both the orientation θ and the orientation $\theta + 45°$ of a Nicol prism in two arrangements. We thus may use any representation $\vec{i}(\theta) = (i_1(\theta), i_2(\theta))$ of the catalog of our knowledge about two mutually complementary propositions $\{\mathcal{P}_1(\theta), \mathcal{P}_2(\theta)\}$ to specify the total information content of the photon (Fig. 2)

$$I_{total} = I_1(\theta) + I_2(\theta) = i_1^2(\theta) + i_2^2(\theta) = 1 \qquad \forall\theta. \tag{7}$$

5 Malus' Law in Quantum Mechanics

The property of the invariance of the total information carried by the photon's polarization implies that with gradually changed experimental parameter from θ_0 to θ the information vector rotates with conservation of the length of the information vector

$$\vec{i}(\theta) = \hat{R}(\theta - \theta_0, \theta_0)\vec{i}(\theta_0), \tag{8}$$

where $\hat{R}(\theta - \theta_0, \theta_0)$ is a rotation matrix $\hat{R}^{-1}(\theta - \theta_0, \theta_0) = \hat{R}^T(\theta - \theta_0, \theta_0)$. Equation (8) expresses our expectation that the transformation law is independent of the actual information vector transformed.

[2] Clearly, the full set of mutually complementary propositions includes also the proposition: "The polarization of the photon is left (right) circular" with associated information i_3, and $I_{total} = I_1 + I_2 + I_3 = i_1^2 + i_2^2 + i_3^2$. Here we may restrict our analysis to the two mutually complementary propositions given above.

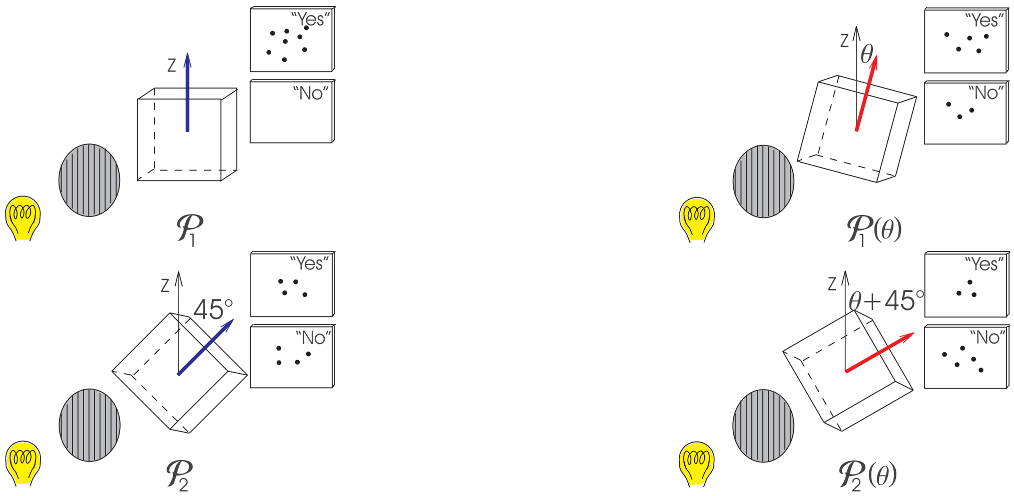

Fig. 2. Two different sets $\{\mathcal{P}_1, \mathcal{P}_2\}$ and $\{\mathcal{P}_1(\theta), \mathcal{P}_2(\theta)\}$ of mutually complementary polarization measurements. They correspond to the following two sets of mutually complementary propositions {"The polarization of the photon is vertical", "The polarization of the photon is $+45°$"} and {"The polarization of the photon is $+\theta$", "The polarization of the photon is $+(\theta + 45°)$"}, respectively. The total information carried by the photon's polarization is independent of the particular set of mutually complementary propositions considered, i.e. $I_{total} = I_1 + I_2 = 1 = I_1(\theta) + I_2(\theta)$.

We assume that no physical process distinguishes one specific orientation of the Nicol prism from others, that is, that the parametric θ-axis is *homogeneous*. This is equivalent with the assumption that the ordinary space is *isotropic*. Suppose two sets of mutually complementary experimental arrangements are associated to a specific parametric value θ_0 and to some other value $\theta_0 + b$ $(-\infty < b < +\infty)$ respectively. Furthermore, suppose the information vectors $\vec{i}(\theta_0)$ and $\vec{i}(\theta_0 + b)$ associated to the two sets of complementary experiments are equal (i.e. all components of the two vectors are equal). If we change the physical parameter in each experiment in the two sets of complementary experiments by an equal interval of $\theta - \theta_0$, the homogeneity of the parametric θ-axis requires that the resulting information vectors will be equivalent. Formally, if for all θ_0, $\vec{i}(\theta_0) = \vec{i}(\theta_0 + b)$ implies $\hat{R}(\theta - \theta_0, \theta_0)\vec{i}(\theta_0) = \hat{R}(\theta - \theta_0, \theta_0 + b)\vec{i}(\theta_0 + b)$ then

$$\hat{R}(\theta - \theta_0, \theta_0) = \hat{R}(\theta - \theta_0, \theta_0 + b). \tag{9}$$

The transformation matrix $\hat{R}$ then depends only on the difference between initial and final value of the experimental parameter and not on the location of these values on the parametric θ-axis.

The orthogonality condition leads to the following general form of the transformation matrix

$$\hat{R}(\theta) = \begin{pmatrix} f(\theta) & -g(\theta) \\ g(\theta) & f(\theta) \end{pmatrix}, \tag{10}$$

where $f(\theta)$ and $g(\theta)$ are not yet specified but differentiable functions satisfying

$$f^2(\theta) + g^2(\theta) = 1, f(0) = 1, \text{ and } f(\pi/4) = 0. \tag{11}$$

Here we take $\theta_0 = 0$ for simplicity.

A change of the experimental parameter in two mutually complementary arrangements from 0 to θ and subsequently from θ to $\theta + d\theta$ must have the same physical effect as a direct change of the parameter from 0 to $\theta + d\theta$. The resulting transformation is therefore independent whether we apply two consecutive transformations $\hat{R}(\theta)$ and $\hat{R}(d\theta)$ or a single transformation $\hat{R}(\theta + d\theta)$, i.e.

$$\hat{R}(\theta + d\theta) = \hat{R}(\theta)\hat{R}(d\theta). \tag{12}$$

Inserting the form (10) of the transformation matrix into the latter expression, one obtains

$$f(\theta + d\theta) = f(\theta)f(d\theta) - g(\theta)g(d\theta). \tag{13}$$

Using the first condition in Eq.(11), we now may transform Eq.(13) into the differential equation

$$\frac{df(\theta)}{d\theta} = -g'(0)\sqrt{1 - f^2(\theta)}. \tag{14}$$

With integration limits defined by Eq.(11) the solution of the differential equation reads

$$f(\theta) = \cos 2\theta. \tag{15}$$

This finally leads to

$$\hat{R}(\theta) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{16}$$

This result directly leads to the familiar expression $p = \cos^2\theta$ for probability in quantum theory. The value θ is directly related to the relative orientation between the Nicol prism and the polarization vector of the photon, or to the phase shift between two paths inside the interferometer in the interference experiment with a Mach-Zehnder type of interferometer [6] or to the relative orientation between the measurement direction and the spin vector of the spin-1/2 particle in the Stern-Gerlach experiment [8].

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