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CONDITIONAL QUANTUM STATE ENGINEERING AT BEAM SPLITTER ARRAYS*

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The generation of arbitrary single-mode quantum states from the vacuum by alternate coherent displacement and photon adding as well as the measurement of the overlap of a signal with an arbitrarily chosen quantum state are studied. With regard to implementations, the transformation of the quantum state of a traveling optical field at an array of beam splitters is considered, using conditional measurement. Allowing for arbitrary quantum states of both the input reference modes and the output reference modes on which the measurements are performed, the setup is described within the concept of two-port non-unitary transformation, and the overall non-unitary transformation operator is derived. It is shown to be a product of operators, where each operator is assigned to one of the beam splitters and can be expressed in terms of an s-ordered operator product, with s being determined by the beam splitter transmittance or reflectance. As an example we discuss the generation of and overlap measurement with Schrödinger-cat-like states.

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1 Introduction

If two traveling (pulse-shaped) modes of the radiation field are mixed at a beam splitter, then the two outgoing modes are in an entangled state in general. Therefore, the reduced state of one of them depends on the result of a measurement performed on the other. By mixing a signal pulse with a reference pulse prepared in a known state and discriminating from all pulses leaving the signal output port those corresponding to a particular measurement result in the other output port, quantum state engineering can be realized [1–3]. On the other hand, the overlap of a signal with a chosen quantum state may be obtained by mixing the signal mode with a reference mode prepared in a quantum state that is specific to the overlap and performing a measurement on the outgoing field [4]. Hence, novel possibilities of direct quantum state generation and measurement are offered, provided that the designed reference states can be prepared and the required measurements can be realized.

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In what follows we present a scheme for the generation of arbitrary quantum states of traveling fields, in which coherent and 1-photon Fock states are fed into an array of beam splitters and zero-photon measurements are performed. We then show how the scheme can be modified in order to measure the overlap of an unknown quantum state of a signal mode with an arbitrarily chosen quantum state. Whereas in the former case a source for 1-photon Fock states should be available, in the latter case only 1-photon Fock state detection is required.

In Section 2 the underlying formalism is outlined and the basic formulas are given. The problem of the generation of arbitrary quantum states is considered in Section 3, and Section 4 is devoted to the problem of overlap measurements. In order to give an example, we consider in Section 5 the generation of and measurement of overlap with Schrödinger-cat-like states. A summary and some concluding remarks are given in Section 6.

2 Conditional quantum state transformation

Let us consider the state transformation at a beam splitter array. As outlined in Fig. 1, the

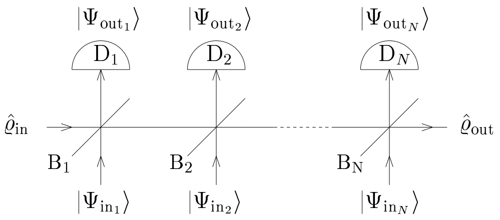

Fig. 1. Conditional quantum state manipulation at an array of beam splitters B_k. The incoming signal state $\hat{\varrho}_{\mathrm{in}}$ is combined with states $|\Psi_{\mathrm{in}_k}\rangle$, and $\hat{\varrho}_{\mathrm{out}}$ is the signal state generated under the condition that the outgoing reference modes have been measured to be in states $|\Psi_{\mathrm{out}_k}\rangle$ by means of the measuring devices D_k.

incoming signal prepared in a state $\hat{\varrho}_{\mathrm{in}}$ passes an array of N beam splitters $\mathrm{B}_1,\ldots,\mathrm{B}_N$ at which it is mixed with reference input modes in states $|\Psi_{\mathrm{in}_1}\rangle,\ldots,|\Psi_{\mathrm{in}_N}\rangle$. When the measuring devices $\mathrm{D}_1,\ldots,\mathrm{D}_N$ detect the reference output modes in states $|\Psi_{\mathrm{out}_1}\rangle,\ldots,|\Psi_{\mathrm{out}_N}\rangle$, then the conditional signal output state reads

$$\hat{\varrho}_{\mathrm{out}} = \frac{1}{p}\hat{Y}\hat{\varrho}_{\mathrm{in}}\hat{Y}^{\dagger}, \tag{1}$$

where

$$p = \mathrm{Tr}\left(\hat{Y}\hat{\varrho}_{\mathrm{in}}\hat{Y}^{\dagger}\right) \tag{2}$$

is the probability of generating $\hat{\varrho}_{\mathrm{out}}$.

The non-unitary transformation operator

$$\hat{Y} = \hat{Y}_N \cdots \hat{Y}_2\hat{Y}_1 \tag{3}$$

is the product of the individual conditional operators

$$\hat{Y}_k = \langle\Psi_{\mathrm{out}_k}|\hat{U}_k|\Psi_{\mathrm{in}_k}\rangle\,, \tag{4}$$

where

$$\hat{U}_k = e^{\mathrm{i}(\varphi_T+\varphi_R)\hat{L}_z}\, e^{2\mathrm{i}\vartheta\hat{L}_y}\, e^{\mathrm{i}(\varphi_T-\varphi_R)\hat{L}_z} = T^{\hat{n}}\, e^{-R^*\hat{a}_k^{\dagger}\hat{a}}\, e^{R\hat{a}^{\dagger}\hat{a}_k} T^{-\hat{n}_k} \tag{5}$$

is the unitary transformation operator of the beam splitter B_k in Fig. 1, with $\hat{L}_y = \mathrm{i}(\hat{a}_k^\dagger \hat{a} - \hat{a}^\dagger \hat{a}_k)/2$ and $\hat{L}_z = (\hat{n} - \hat{n}_k)/2$ [5, 6]. Here, $T = \cos\vartheta e^{\mathrm{i}\varphi_T}$ and $R = \sin\vartheta e^{\mathrm{i}\varphi_R}$ are the transmittance and reflectance, respectively. The operators $\hat{Y}_k$ can be represented as s-ordered operator products [3]

$$\hat{Y}_k = \left\{ \hat{F}(R\hat{a}^\dagger)\, \hat{G}^\dagger\left(-\frac{R}{T^*}\hat{a}^\dagger\right)\right\}_s T^{\hat{n}}, \tag{6}$$

where the operators $\hat{F}$ and $\hat{G}$, respectively, generate $|\Psi_{\mathrm{in}_k}\rangle$ and $|\Psi_{\mathrm{out}_k}\rangle$ from the vacuum,

$$|\Psi_{\mathrm{in}_k}\rangle = \hat{F}(\hat{a}_k^\dagger)|0\rangle_k\,, \quad |\Psi_{\mathrm{out}_k}\rangle = \hat{G}(\hat{a}_k^\dagger)|0\rangle_k\,, \tag{7}$$

and the ordering parameter s is determined by the absolute value of the beam splitter reflectance as

$$s = \frac{2}{|R|^2} - 1. \tag{8}$$

Note that the ordering procedure in (6) can be omitted if $|\Psi_{\mathrm{in}_k}\rangle$ or $|\Psi_{\mathrm{out}_k}\rangle$ is a coherent state [3], since for

$$|\Psi_{\mathrm{in}_k}\rangle = \hat{D}_k(\alpha)\hat{F}(\hat{a}_k^\dagger)|0\rangle_k\,, \quad |\Psi_{\mathrm{out}_k}\rangle = \hat{D}_k(\beta)\hat{G}(\hat{a}_k^\dagger)|0\rangle_k \tag{9}$$

we have

$$\hat{Y}_k = \hat{D}\left(\frac{\alpha - T\beta}{R^*}\right)\, \hat{Y}_k\binom{\alpha=0}{\beta=0}\, \hat{D}\left(\frac{\beta - T^*\alpha}{R^*}\right). \tag{10}$$

3 Generation of truncated quantum states $|\Psi\rangle$

Each quantum state $|\Psi\rangle$ that is composed of a finite number of Fock states $|n\rangle$ can be written as

$$|\Psi\rangle = \sum_{n=0}^{N} \psi_n |n\rangle = \frac{\psi_N}{\sqrt{N!}} \prod_{k=1}^{N} (\hat{a}^\dagger - \beta_k^*)|0\rangle, \tag{11}$$

where $\beta_1, \ldots, \beta_N$ denote the N solutions of the equation $\langle\Psi|\beta\rangle \equiv \langle\Psi|\hat{D}(\beta)|0\rangle = 0$. This reveals that $|\Psi\rangle$ can be generated from the vacuum state by alternate displacement and photon adding, as outlined in Fig. 2 [2]. A coherent displacement $\hat{D}(\alpha)$ can be realized by mixing the

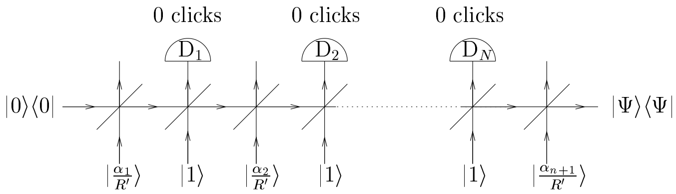

Fig. 2. Generation of truncated states $|\Psi\rangle$ from the vacuum by alternate displacement and photon adding. The coherent displacements are achieved by combination with highly excited coherent states $|\alpha/R'\rangle$ at highly transmitting beam splitters ($T' \to 1$), and photon adding is realized by combination with one-photon Fock states $|1\rangle$ and measuring zero photons in the outgoing reference modes with photodetectors D_k.

mode with a reference mode in a strong coherent state $|\alpha/R'\rangle$ at a highly transmitting beam splitter ($T' \to 1$) [7], and photon adding is achieved by mixing the mode with a reference mode in a Fock state $|1\rangle$ and measuring zero photons in the output detection channel (detectors D_k in Fig. 2) [6].

We assume that all the beam splitters used for photon adding have the same transmittance T and reflectance R. The non-unitary transformation operator then reads as

$$\hat{Y} = \hat{D}(\alpha_{N+1})\hat{Y}_N\hat{D}(\alpha_N)\cdots\hat{Y}_1\hat{D}(\alpha_1)\,, \tag{12}$$

where

$$\hat{Y}_k = R\hat{a}^\dagger T^{\hat{n}}\,, \tag{13}$$

and the complex parameters $\alpha_1,\ldots,\alpha_{N+1}$ are determined from the equation

$$\frac{\hat{Y}|0\rangle\langle 0|\hat{Y}^\dagger}{||\hat{Y}|0\rangle||^2} = |\Psi\rangle\langle\Psi| \tag{14}$$

as $\alpha_k = T^{*N+1-k}(\beta_{k-1}-\beta_k)$ for $k = 2,\ldots,N+1$ ($\beta_{N+1}=0$), and $\alpha_1 = -\sum_{l=1}^{N} T^{-l}\alpha_{l+1}$. The probability of generating the desired state $\hat{\varrho}_{\mathrm{out}} = |\Psi\rangle\langle\Psi|$, i.e. the probability that all N detectors register zero photons, is given by

$$||\hat{Y}|0\rangle||^2 = \frac{N!}{|\psi_N|^2}\frac{|R|^{2N}}{|T|^{N(1-N)}}\exp\left[-|R|^2\sum_{k=1}^{N}\left|\frac{\sum_{l=1}^{k}|T|^{2l}(\beta_{N+2-l}-\beta_{N+1-l})}{T^{k+2}}\right|^2\right] \tag{15}$$

and decreases rapidly with increasing N. Nevertheless, for small N this scheme offers a way to generate specific traveling quantum states, given the possibility to prepare 1-photon Fock states.

4 Measuring arbitrary overlaps

One fundamental task in quantum mechanics is to find the probability that a specific quantum state $|\Psi\rangle$ is contained in a state $\hat{\varrho}_{\mathrm{in}}$ of a given system. With regard to traveling waves, this overlap $\langle\Psi|\hat{\varrho}_{\mathrm{in}}|\Psi\rangle$ can be measured for a given state $|\Psi\rangle$ of the type (11) as outlined in Fig. 3 [8]. The signal mode in state $\hat{\varrho}_{\mathrm{in}}$ is mixed with reference modes in coherent states $|\alpha_1\rangle,\ldots,$

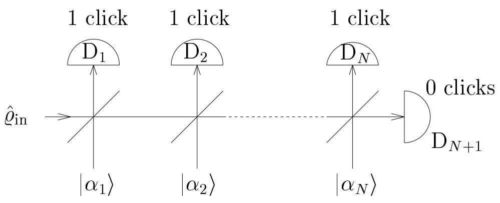

Fig. 3. Measurement of the overlap $\langle\Psi|\hat{\varrho}_{\mathrm{in}}|\Psi\rangle$ of a signal state $\hat{\varrho}_{\mathrm{in}}$ with a desired state $|\Psi\rangle$ by successive combination with appropriately chosen coherent states $|\alpha_1\rangle,\ldots,|\alpha_N\rangle$ at beam splitters and measurement of the relative frequency of detecting simultaneously 1 photon at the photodetectors $\mathrm{D}_1,\ldots,\mathrm{D}_N$ and 0 photons at D_{N+1}.

$|\alpha_N\rangle$ at an array of N beam splitters. Photodetectors $\mathrm{D}_1,\ldots,\mathrm{D}_{N+1}$ perform photon number measurements at the output modes. The joint probability $p(1,1;2,1;\ldots;N,1;N+1,0)$ that each of the detectors $\mathrm{D}_1,\ldots,\mathrm{D}_N$ registers one photon and D_{N+1} none is then given by

$$p(1,1;2,1;\ldots;\ N,1;\ N+1,0) = \langle 0|\hat{Y}\hat{\varrho}_{\mathrm{in}}\hat{Y}^\dagger|0\rangle \tag{16}$$

with

$$\hat{Y} = \hat{Y}_N \cdots \hat{Y}_2 \hat{Y}_1 \,, \tag{17}$$

where for identical beam splitters

$$\hat{Y}_k = -R^* \hat{D}\left(\frac{\alpha_k}{R^*}\right) T^{\hat{n}} \hat{a} \hat{D}\left(-\frac{T^*}{R^*}\alpha_k\right), \tag{18}$$

see (4) as well as (6) and (10). This expression reveals that the signal is manipulated by alternate displacement and photon subtraction. If we now choose the arguments α_k ($k=1,\ldots,N$) of the coherent states as $\alpha_k = (R^*/T^{*k}) \sum_{l=1}^{k} |T|^{2l-1}(\beta_l - \beta_{l-1})$, with $\beta_0 = 0$, so that for a chosen $|\Psi\rangle$ the relation

$$\frac{\hat{Y}^\dagger|0\rangle\langle 0|\hat{Y}}{\|\hat{Y}^\dagger|0\rangle\|^2} = |\Psi\rangle\langle\Psi| \tag{19}$$

is valid, then from (16) it is seen that the sought overlap can be obtained from the joint probability as

$$\langle\Psi|\hat{\varrho}_{\rm in}|\Psi\rangle = \frac{p(1,1;2,1;\ldots;N,1;N+1,0)}{\|\hat{Y}^\dagger|0\rangle\|^2} \,. \tag{20}$$

The denominator $\|\hat{Y}^\dagger|0\rangle\|^2$ is given by (15), with $(\beta_{N+2-l} - \beta_{N+1-l})$ being replaced with $(\beta_l - \beta_{l-1})$. It may be regarded as being the “fidelity” of the measurement, since it is a measure of the maximal occurence of the detection coincidences. Obviously, it is equal to $p(1,1;2,1;\ldots;N,1;N+1,0)$ in the particular case when signal and measured state coincide, $\hat{\varrho}_{\rm in} = |\Psi\rangle\langle\Psi|$.

5 Schrödinger-cat-like states

The schemes in Figs. 2 and 3 can be simplified if some of the β_k in (11) are equal:

$$|\Psi\rangle = \frac{\psi_N}{\sqrt{N!}} \prod_{l=1}^{M} (\hat{a}^\dagger - \beta_l^*)^{d_l} |0\rangle \tag{21}$$

with $M < N$. In this case it is possible to add d_k photons at once in Fig. 2 by using M detectors and combining with Fock states $|d_k\rangle$ or to subtract d_k photons at each beam splitter in Fig. 3 by using $M+1$ detectors and measuring the relative frequency of the event $(1,d_1;\ldots;M,d_M;M+1,0)$. All calculations are analogous if $R\hat{a}^\dagger$ in (13) is replaced with $(R\hat{a}^\dagger)^{d_k}/\sqrt{d_k!}$ and $-R^*\hat{a}$ in (18) is replaced with $(-R^*\hat{a})^{d_k}/\sqrt{d_k!}$.

As an example let us consider the states

$$|\Psi_n^{\alpha,\beta}\rangle = \frac{1}{\sqrt{\mathcal{N}}} \hat{D}(\gamma_3)\hat{a}^{\dagger n}\hat{D}(\gamma_2)\hat{a}^{\dagger n}\hat{D}(\gamma_1)|0\rangle, \tag{22}$$

where $\gamma_1 = {\rm i}(\beta-\alpha)/2$, $\gamma_2 = {\rm i}(\alpha-\beta)$, $\gamma_3 = [(1-{\rm i})\alpha + (1+{\rm i})\beta]/2$, and the normalization factor is $\mathcal{N} = (4^n n!/\sqrt{\pi})\Gamma(n+1/2)\,{}_1F_2[-n,1/2-n,1,|\alpha-\beta|^4/64]$. From the above considerations it is clear that these states can be generated using 2 detectors, and the overlap of a signal state with such states can be measured using 3 detectors. The states $|\Psi_n^{\alpha,\beta}\rangle$ reveal the interesting property that for increasing $n = |\alpha - \beta|^2/4$ they approach superpositions of coherent states [8], $|\Psi_\infty^{\alpha,\beta}\rangle\langle\Psi_\infty^{\alpha,\beta}| = |\Psi^{\alpha,\beta}\rangle\langle\Psi^{\alpha,\beta}|$ with

$$|\Psi^{\alpha,\beta}\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle + |\beta\rangle\right) \tag{23}$$

(note that $|\langle\Psi_{n=3}^{\alpha,\beta}|\Psi^{\alpha,\beta}\rangle|^2 > 0.95$). This offers the possibility of generating Schrödinger-cat-like states, provided that two n-photon Fock states are available. On the other hand, measuring overlaps with Schrödinger-cat-like states allows one, in principle, to reconstruct the signal state [9].

It is worth noting that choosing a squeezed coherent signal state offers the possibility of measuring the field strength statistics of a Schrödinger cat without need to generate the cat state. Note that squeezed coherent states approach field strength states for sufficiently strong squeezing.

Finally, it should be pointed out that the probability of generation of the states $|\Psi_n^{\alpha,\beta}\rangle$ and the fidelity of measurement of the overlap with them are equal. For large n, they decrease exponentially with increasing n:

$$p = \frac{|2R^2T|^{2n}}{n\pi}\,\exp\left[n\left(1-\left|\frac{R}{T}\right|^2(1+|T|^{-2}(1-2|T|^2)^2)\right)\right]. \tag{24}$$

6 Conclusion

We have discussed conditional quantum state engineering at beam splitter arrays. We have presented a scheme for the generation of arbitrary quantum states which requires coherent states and 1-photon Fock states and zero-photon detection. Further, we have given a scheme for the measurement of the overlap of an unknown signal state with an arbitrary quantum state which requires coherent states and 0- and 1-photon detections. For the two schemes we have calculated the probabilities of state generation and overlap measurement. Finally, we have shown how the schemes can be simplified under special conditions. As an example we have considered the generation of (and overlap measurement with) states which approach superpositions of two arbitrary coherent states.

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