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BERRY PHASE DUE TO QUANTUM MEASUREMENTS*

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The usual, "static" version of the quantum Zeno effect consists in the hindrance of the evolution of a quantum systems due to repeated measurements. There is however a "dynamic" version of the same phenomenon, first discussed by von Neumann in 1932 and subsequently explored by Aharonov and Anandan, in which a system is forced to follow a given trajectory. A Berry phase appears if such a trajectory is a closed loop in the projective Hilbert space. A specific example involving neutron spin is considered and a similar situation with photon polarization is investigated.

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1 Introduction

The usual, *static* version of the quantum Zeno effect (QZE) consists in hindering (and eventually halting) the time evolution of a quantum system by repeatedly checking if it has decayed [1]. In a few words, this is due to the fact that in time dt, by the Schrödinger equation, the phase of a state $\psi(t)$ changes by $\mathrm{O}(dt)$ while the absolute value of its scalar product with the initial state changes by $\mathrm{O}(dt^2)$. The *dynamic* quantum Zeno effect exploits the above features and forces the evolution through an arbitrary trajectory by a series of repeated measurements [2, 3]: Let there be a family of states ϕ_k, $k = 0, 1, \dots, N$, such that $\phi_0 = \psi(0)$, and such that successive states differ little from one another (i.e., $|\langle\phi_{k+1}|\phi_k\rangle|$ is nearly 1). Now let $\delta T = T/N$ and at $T_k = k\delta T$ project the evolving wave function on ϕ_k. Then for sufficiently large N, $\psi(T) \approx \phi_N$. [The static QZE is the special case $\phi_k = \phi_0 (= \psi(0))\ \forall\, k$.]

In the following we will show how guiding a system through a closed loop in its state space (projective Hilbert space) leads to a geometrical phase [3–5]. We will first summarize some results valid for neutron spin [6, 7] and then consider the case of photon polarization [8].

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2 Neutron spin

Assume first that there is *no* Hamiltonian acting on the system: the neutron crosses a region where no magnetic field is present. It starts with spin up along the z-axis and is projected on the family of states

$$\phi_k \equiv \exp(-i\theta_k \boldsymbol{\sigma}\cdot\boldsymbol{n}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{with } \theta_k \equiv \frac{ak}{N} \,, \qquad k = 0, \ldots, N \,, \tag{1}$$

where $\boldsymbol{\sigma}$ is the vector of the Pauli matrices and $\boldsymbol{n} = (n_x, n_y, n_z)$ a unit vector (independent of k).

The neutron evolves for a time T with projections at times $T_k = k\delta T$ ($k = 1, \ldots, N$ and $\delta T = T/N$). The final state is $\left[\phi_0 = \binom{1}{0}\right]$

$$\begin{aligned} |\psi(T)\rangle &= |\phi_N\rangle\langle\phi_N|\phi_{N-1}\rangle \cdots \langle\phi_2|\phi_1\rangle\langle\phi_1|\phi_0\rangle \\ &= \cos^N\left(\frac{a}{N}\right)\left(1 + in_z \tan\frac{a}{N}\right)^N |\phi_N\rangle \\ &\stackrel{N\to\infty}{\longrightarrow} \exp(ian_z)\exp(-ia\boldsymbol{\sigma}\cdot\boldsymbol{n})|\phi_0\rangle. \end{aligned} \tag{2}$$

If $a = \pi$,

$$\psi(T) = \exp[-i\pi(1-\cos\Theta)]\phi_0 = \exp(-i\Omega/2)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3}$$

where $\cos\Theta \equiv n_z$ and Ω is the solid angle subtended by the curve traced by the spin during its evolution. The factor $\exp(-i\Omega/2)$ is a Berry phase and it is due only to measurements (the Hamiltonian is zero). Notice that, as discussed by Pati and Lawande [9], no Berry phase appears in the usual quantum Zeno context, namely when $\phi_k \propto \phi_0\ \forall\, k$, because in that case $a = 0$ in (2).

We now look at the process (2) for N finite. The spin goes back to its initial state after describing a regular polygon on the Poincaré sphere, as in Fig. 1(a). After $N(<\infty)$ projections the final state is [7]

$$\psi(T) = \rho_N \exp(-i\beta_N)\phi_0, \tag{4}$$

where

$$\rho_N = \left(\cos^2\frac{\pi}{N} + n_z^2\sin^2\frac{\pi}{N}\right)^{\frac{N}{2}}, \qquad \beta_N = \pi - N\arctan\left(\cos\Theta\tan\frac{\pi}{N}\right). \tag{5}$$

The quantity ρ_N accounts for the probability loss (N is finite and there is no QZE). It is easy to check that in the "continuous measurement" limit (QZE) we recover the result (3).

The relation between the solid angle and the geometrical phase is valid also with a finite number of polarizers N. Indeed, it is straightforward to show that the solid angle subtended by a regular N-sided polygon [Fig. 1(a)] is

$$\Omega_N = 2\pi - 2N\arctan\left(\cos\Theta\tan\frac{\pi}{N}\right) = 2\beta_N. \tag{6}$$

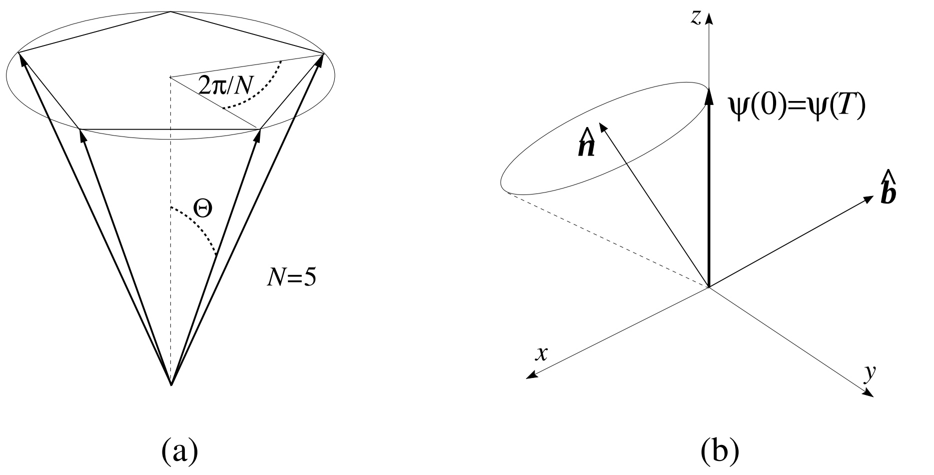

Fig. 1. (a) Spin evolution due to $N = 5$ measurements. (b) Spin evolution with very frequent measurements and non-zero Hamiltonian.

This result is of course in agreement with other analyses [10] based on the Pancharatnam connection [4].

Let us now consider the effect of a non-zero Hamiltonian (neutron spin in a magnetic field)

$$H = \mu \boldsymbol{\sigma} \cdot \boldsymbol{b}, \tag{7}$$

where $\boldsymbol{b} = (b_x, b_y, b_z)$ is a unit vector, in general different from $\boldsymbol{n}$. See Fig. 1(b).

If the system starts with spin up it has the following "undisturbed" evolution

$$\psi(t) = \exp(-i\mu t \boldsymbol{\sigma} \cdot \boldsymbol{b})\phi_0. \tag{8}$$

Now let the system evolve for a time T with projections at times $T_k = k\delta T$ ($k = 1, \dots, N$ and $\delta T = T/N$) and Hamiltonian evolution in between. It is not difficult to show that, in the continuum limit ($N \to \infty$), the final state reads [7]:

$$\psi(T) = \exp\left(-i\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt\right) \exp\left(ian_z - ia\boldsymbol{\sigma} \cdot \boldsymbol{n}\right)\phi_0. \tag{9}$$

The first factor in (9) is obviously the dynamical phase and the remaining phase, when the spin goes back to its initial state, is the geometrical phase: when $a = \pi$

$$\psi(T) = \exp\left(-i\Omega/2\right) \exp\left(-i\mu T(\boldsymbol{b} \cdot \boldsymbol{n})n_z\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{10}$$

where Ω is the solid angle subtended by the curve traced out by the spin, as in Eq. (3), and $\mu T(\boldsymbol{b} \cdot \boldsymbol{n})n_z$ is the dynamical phase.

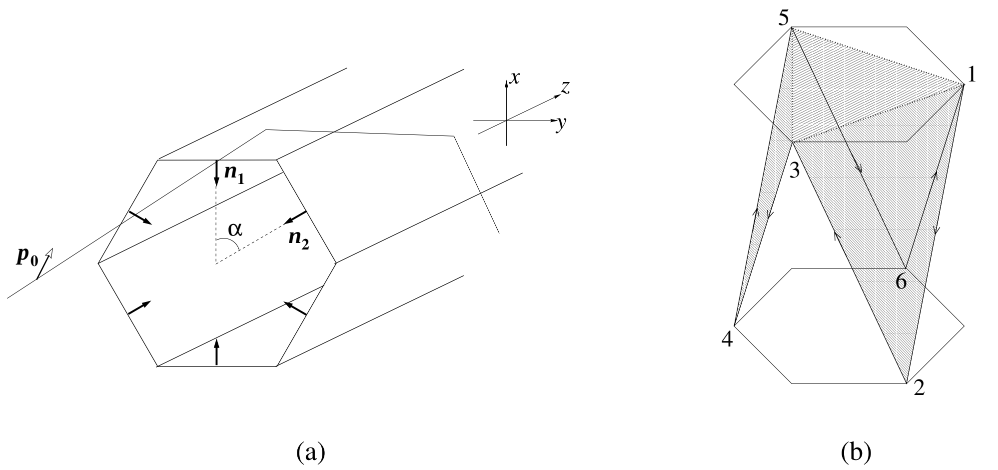

Fig. 2. (a) Klein's arrangement: a photon is sent into a polygonal cylinder, bouncing off $N = 6$ perfect mirrors and emerging from the opposite side. (b) Evolution of the polarization. The points 1-6 represent the tip of the polarization vector and the shadow area yields the Berry phase.

3 Photon polarization

The experiments described in Sec. 2 involve neutrons, or in general *massive* spin-$1/2$ particles. This allowed us to neglect the neutron momentum in our analysis: the axis of the neutron polarizer can have an arbitrary direction with respect to the neutron momentum. Equations very similar to those of Sec. 2 were obtained by Berry and Klein in their beautiful work on polarized light [8]. However, when one deals with photons, the additional constraint of transversality makes things more complicated: the photon polarization must be perpendicular to momentum and for this reason, in Ref. [8], the projection on states of non-linear polarization is achieved by making use of "retarders" [11].

It is interesting, in this context, to discuss a geometric configuration proposed by A.G. Klein [12]. A photon is sent into a polygonal cylinder made up of N perfect plane mirrors and emerges after N reflections: see Fig. 2 ($N = 6$). Let us consider a photon with momentum parallel to $\hat{\boldsymbol{k}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and polarization $|p\rangle$, which is reflected by an ideal mirror with normal unit vector $\boldsymbol{n}$. After reflection, the photon helicity is reversed and the momentum becomes

$$\hat{\boldsymbol{k}}' = -R_{\boldsymbol{n}}(\pi)\hat{\boldsymbol{k}}, \tag{11}$$

where $R_{\boldsymbol{n}}(\phi)$ represents a rotation by an angle ϕ around direction $\boldsymbol{n}$. Hence, after a reflection, the polarization state becomes (ideal mirror, infinite conductivity)

$$|p'\rangle = M(\boldsymbol{n})|p\rangle, \quad \text{with} \quad M(\boldsymbol{n}) = \exp\left(-i\boldsymbol{n}\cdot\boldsymbol{J}\pi\right), \tag{12}$$

where $\boldsymbol{J} = (J_x, J_y, J_z)$ are the generators of rotations in $\mathbf{R}^3$. Consider now a polygonal cylinder of ideal mirrors, whose normal vectors are

$$\boldsymbol{n}_1 = (-1, 0, 0), \quad \boldsymbol{n}_\ell = R_z[(\ell-1)\alpha]\boldsymbol{n}_1 = [-\cos(\ell-1)\alpha, -\sin(\ell-1)\alpha, 0], \tag{13}$$

where $\alpha = 2\pi/N$ for a regular N-sided polygon [see Fig. 2(a)]. The operator representing the action of the ℓ-th mirror is $M_\ell = M(\boldsymbol{n}_\ell)$. By using (12) and (13) one gets

$$M_\ell = -\exp\left(-i\alpha(\ell-1)J_z\right) M_1 \exp\left(i\alpha(\ell-1)J_z\right), \quad M_1 = \exp(i\pi J_x). \tag{14}$$

Let $|p_0\rangle$ be the initial photon polarization; after N reflections, the photon emerges with final polarization

$$|p_N\rangle = M_N \cdots M_2 M_1 |p_0\rangle = \exp\left(-iN\alpha J_z\right) \widetilde{M}^N |p_0\rangle, \tag{15}$$

where

$$\widetilde{M} = \exp\left(i\alpha J_z\right) M_1 = \exp\left(i\alpha J_z\right) \exp(i\pi J_x). \tag{16}$$

By using $[J_i, J_j] = i\varepsilon_{ijk} J_k$, we obtain

$$\widetilde{M} = \exp\left(-i\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J}\right), \quad \text{with} \quad \tilde{\boldsymbol{n}} = \left(-\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}, 0\right), \tag{17}$$

so that $\widetilde{M}$ is a π rotation around $\tilde{\boldsymbol{n}}$. Remembering that $N\alpha = 2\pi$ and using (17), the final polarization (15) reads

$$|p_N\rangle = \exp(-i2\pi J_z) \exp\left(-iN\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J}\right) |p_0\rangle = \exp\left(-iN\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J}\right) |p_0\rangle. \tag{18}$$

For odd $N = 2m+1$, one gets

$$|p_N\rangle = \exp\left(-i(2m+1)\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J}\right) |p_0\rangle = \exp\left(-i\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J}\right) |p_0\rangle \tag{19}$$

and the final polarization is not in the same direction as the initial one. On the other hand, for even $N = 2m$, the polarization vector does describe a closed loop, but one obtains

$$|p_N\rangle = \exp(-i2m\pi\tilde{\boldsymbol{n}} \cdot \boldsymbol{J})|p_0\rangle = |p_0\rangle \tag{20}$$

and the photon acquires no geometrical phase. The reason for this result is shown in Fig. 2(b): for even N, the polarization vector *always* describes a solid angle $\Omega = 2\pi$: the photon always acquires a Berry phase $\beta = \Omega = 2\pi$, with no physical effects.

The experiment just described is not equivalent to the one analyzed in the previous section. Indeed, the dynamics of reflections is always unitary. The difficulty in obtaining a geometrical phase is due to the condition of transversality of the electromagnetic field. To encompass this situation one needs (at least three) mirrors whose normal vectors do not lie in the same plane, as shown in [13].

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