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WIGNER FUNCTION AND COHERENCE PROPERTIES OF COLD AND THERMAL NEUTRONS*

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We analyze the coherence properties of a cold or a thermal neutron by utilizing the Wigner quasidistribution function. We look in particular at a recent experiment performed by Badurek *et al.*, in which a polarized neutron crosses a magnetic field that is orthogonal to its spin, producing highly non-classical states. The quantal coherence is extremely sensitive to the field fluctuation at high neutron momenta. A "decoherence parameter" is introduced in order to get quantitative estimates of the losses of coherence.

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1 Introduction

Highly non-classical, Schrödinger-cat-like neutron states can be produced by coherently superposing different spin states in an interferometer and with neutron spin echo [1, 2]. We analyze here an interesting recent experiment [3] in which a polarized neutron crosses a magnetic field that is orthogonal to its spin, producing Schrödinger-cat-like states. Our main purpose is to investigate the decoherence effects that arise when the fluctuations of the magnetic field are considered.

2 Squeezing and squashing

Let us start by looking at the coherence properties of a neutron wave packet and concentrate our attention on the losses of coherence provoked by a fluctuating magnetic field. To this end, we introduce the Wigner quasidistribution function

$$W(x,k) = \frac{1}{2\pi} \int d\xi \; e^{-ik\xi} \psi\left(x + \frac{\xi}{2}\right) \psi^*\left(x - \frac{\xi}{2}\right), \tag{1}$$

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where x is position, $p = \hbar k$ momentum and ψ the wave function of the neutron in the apparatus. The Wigner function is normalized to one and its marginals represent the position and momentum probability distributions

$$\int dx\; dk\; W(x,k) = 1; \qquad P(x) = \int dk\; W(x,k), \quad P(k) = \int dx\; W(x,k). \tag{2}$$

We shall work in one dimension. We assume that the neutron wave function is well approximated by a Gaussian

$$\psi(x) = \frac{1}{(2\pi\delta^2)^{1/4}} \exp\left[-\frac{(x-x_0)^2}{4\delta^2} + ik_0 x\right], \tag{3}$$

$$\begin{aligned} \phi(k) &= \frac{1}{(2\pi\delta_k^2)^{1/4}} \exp\left[-\frac{(k-k_0)^2}{4\delta_k^2} - i(k-k_0)x_0\right] \\ &= \left(\frac{2\delta^2}{\pi}\right)^{1/4} \exp\left[-\delta^2(k-k_0)^2 - i(k-k_0)x_0\right], \end{aligned} \tag{4}$$

where δ is the spatial spread of the wave packet, $\delta_k\delta = 1/2$, x_0 is the initial average position of the neutron and $p_0 = \hbar k_0$ its average momentum. The two functions above are related by a Fourier transformation and are both normalized to one. Normalization will play an important role in our analysis and will never be neglected.

The Wigner function for the state (3), (4) is readily calculated

$$W(x,k) = \frac{1}{\pi} \exp\left[-\frac{(x-x_0)^2}{2\delta^2}\right] \exp\left[-2\delta^2(k-k_0)^2\right] \tag{5}$$

and turns out to be a positive function. In the language of quantum optics [4], we shall say that the neutron is prepared in a coherent state if $\delta = \delta_k = 1/\sqrt{2}$ and in a squeezed state if $\delta \neq \delta_k$. An illustrative example is given in Fig. 1.

Consider now a polarized neutron that crosses a constant magnetic field, parallel to its spin, of intensity B and contained in a region of length L. Since the total energy is conserved, the kinetic energy of the neutron in the field changes by $\Delta E = \mu B > 0$, where $-\mu$ is the neutron magnetic moment. This implies a change in average momentum $\Delta k = m\mu B/\hbar^2 k_0$ and an additional shift of the neutron phase proportional to $\Delta \equiv L\Delta k/k_0$. The resulting effect on the Wigner function is $W(x,k) \to W(x-\Delta,k)$.

Assume now that the intensity of the B-field fluctuates around its average B_0 according to a Gaussian law. This fluctuation is reflected in a fluctuation of the quantity Δ according to the distribution law

$$w(\Delta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\Delta-\Delta_0)^2}{2\sigma^2}\right], \tag{6}$$

where σ is the standard deviation. The ratio σ/Δ_0 is simply equal to the ratio $\delta B/B_0$, δB being the standard deviation of the fluctuating B-field. The average Wigner function, when the neutron has crossed the whole B region of lenght L, represents a "squashed" state, that has partially lost its quantum coherence:

$$W_{\rm m}(x,k) = \int d\Delta\; w(\Delta)\; W(x-\Delta,k). \tag{7}$$

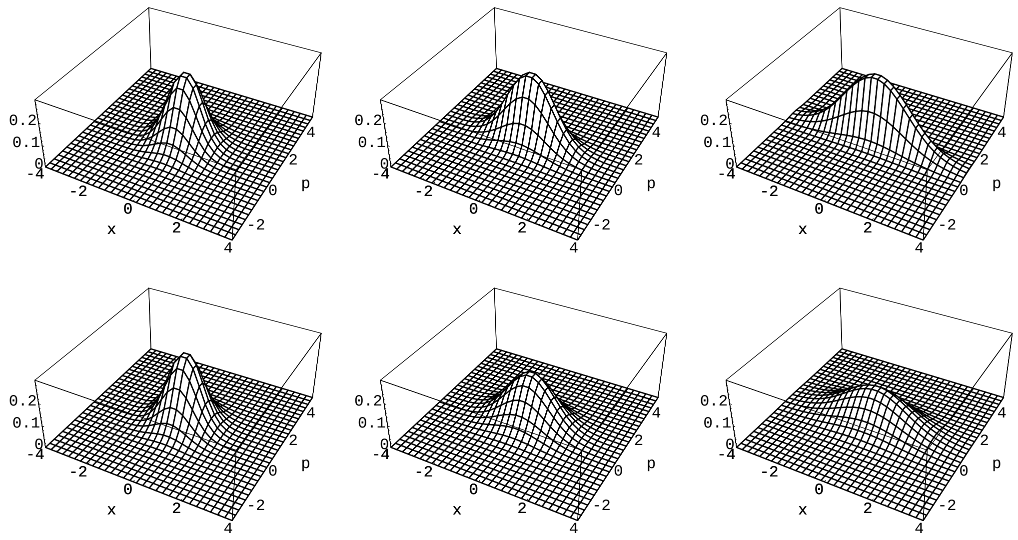

Fig. 1. Above, Wigner function (5), for $x_0 = 0, k_0 = 1.7 \cdot 10^{10}\mathrm{m}^{-1}$; x is in units 10^{-10}m and $p = k$ in units $10^{10}\mathrm{m}^{-1}$. From left to right: $\delta = 1/\sqrt{2}$ (coherent state), $\delta = 1$ (squeezed state) and $\delta = \sqrt{2}$ (a more squeezed state); the uncertainty principle always reads $\delta_k \delta = 1/2$ (minimum uncertainty states). Below, Wigner function in (7) for the same values of x_0, k_0 and $\Delta_0 = 0$. From left to right: $\sigma = 0, 1/\sqrt{2}, \sqrt{3/2}$; the uncertainty principle yields $\delta_k \delta' = 1/2, 1/\sqrt{2}, 1$ (squashing).

This function is represented in Fig. 1 for $\Delta_0 = 0$ (vanishing average magnetic field) and increasing values of σ. The above Wigner function can be calculated explicitly, but its expression is a bit cumbersome; however, its marginals (2) are simple:

$$P(x) = \frac{1}{\sqrt{2\pi(\delta^2+\sigma^2)}} \exp\left[-\frac{(x-x_0-\Delta_0)^2}{2(\delta^2+\sigma^2)}\right], \tag{8}$$

$$P(k) = \sqrt{\frac{2\delta^2}{\pi}} \exp\left[-2\delta^2(k-k_0)^2\right]. \tag{9}$$

Notice that the momentum distribution (9) is unaltered [$|\phi(k)|^2$ in (4)]: obviously, the energy of the neutron does not change. Observe the additional spread in position $\delta' = (\delta^2+\sigma^2)^{1/2}$ and notice that the Wigner function and its marginals are always normalized to one. The uncertainty principle yields $\delta_k \delta' = \frac{1}{2}\sqrt{1+\sigma^2/\delta^2} > 1/2$.

3 Schrödinger-cat states in a fluctuating magnetic field

Let us now look in more detail at the experiment [3]. A polarized ($+y$) neutron enters a magnetic field, perpendicular to its spin, of intensity $B_0 = 0.28$mT, confined in a region of length $L = 57$cm. Due to Zeeman splitting, the two neutron spin states travel with different speeds in the field. The average neutron wavenumber is $k_0 = 1.7 \cdot 10^{10}\mathrm{m}^{-1}$ and its coherence length (defined by a chopper) is $\delta = 1.1 \cdot 10^{-10}$m. By travelling in the magnetic field, the two neutron spin states are separated by a distance $\Delta_0 = 2m\mu B_0/\hbar^2 k_0 = 16.1 \cdot 10^{-10}$m, one order of magnitude larger than δ (notice the factor 2, absent in the definition of the previous section). Observe that the neutron wave packet itself has a natural spread $\delta_t = (\delta^2 + (\hbar t/2m\delta)^2)^{1/2} \simeq 15$cm (due to its free evolution for a time $t \simeq mL/\hbar k_0$); however, we shall neglect this additional effect, because it is irrelevant for the loss of quantum coherence.

After the neutron has crossed the B-field only the $+y$ spin-component is observed and its Wigner function is readily computed

$$
\begin{aligned}
W(x,k) &= \frac{1}{4\pi}\exp[-2\delta^2(k-k_0)^2] \\
&\times \left[\exp\left(-\frac{\left(x-\frac{\Delta}{2}\right)^2}{2\delta^2}\right) + \exp\left(-\frac{\left(x+\frac{\Delta}{2}\right)^2}{2\delta^2}\right) + 2\exp\left(-\frac{x^2}{2\delta^2}\right)\cos(k\Delta)\right].
\end{aligned}
\tag{10}
$$

Notice that for $\Delta = 0$ (no B-field) one obtains (5). Our interest is to investigate the loss of quantum coherence if the intensity of the B-field fluctuates, like in the previous section, yielding a random shift according to the law (6). In such a case, the average Wigner function reads

$$
\begin{aligned}
W_{\rm m}(x,k) &= \int d\Delta\; w(\Delta)\; W(x,k) = \frac{1}{4\pi}\exp[-2\delta^2(k-k_0)^2] \\
&\times \left[\sqrt{\frac{\delta^2}{\delta^2+\frac{\sigma^2}{4}}}\exp\left(-\frac{\left(x-\frac{\Delta_0}{2}\right)^2}{2\left(\delta^2+\frac{\sigma^2}{4}\right)}\right) + \sqrt{\frac{\delta^2}{\delta^2+\frac{\sigma^2}{4}}}\exp\left(-\frac{\left(x+\frac{\Delta_0}{2}\right)^2}{2\left(\delta^2+\frac{\sigma^2}{4}\right)}\right)\right. \\
&\qquad \left. + \; 2\exp\left(-\frac{x^2}{2\delta^2}\right)\exp\left(-\frac{\sigma^2k^2}{2}\right)\cos(k\Delta_0)\right]
\end{aligned}
\tag{11}
$$

and the momentum distribution function yields

$$
P(k) = \sqrt{\frac{\delta^2}{2\pi}}\exp[-2\delta^2(k-k_0)^2]\left[1+\exp\left(-\frac{\sigma^2k^2}{2}\right)\cos(k\Delta_0)\right]. \tag{12}
$$

Notice also that, since only the $+y$-component of the neutron spin is observed, the normalization reads

$$
\begin{aligned}
N &= \int dx\; dk\; W_{\rm m}(x,k) \\
&= \frac{1}{2}\left[1+\sqrt{\frac{\delta^2}{\delta^2+\frac{\sigma^2}{4}}}\exp\left(-\frac{\Delta_0+4\delta^2\sigma^2k_0^2}{8\left(\delta^2+\frac{\sigma^2}{4}\right)}\right)\cos\left(\frac{\delta^2}{\delta^2+\frac{\sigma^2}{4}}k_0\Delta_0\right)\right].
\end{aligned}
\tag{13}
$$

Obviously, $N = 1$ when no magnetic field is present ($\sigma = \Delta_0 = 0$). The Wigner function (11) is plotted in Fig. 2 for some values of σ. The off-diagonal part of the Wigner function ("trustee" of the interference effects) is very fragile at high values of momentum. This was already stressed in [2, 3] and is apparent in the structure of the marginal distribution (12): the term $\exp(-\sigma^2k^2/2)$ strongly suppresses the interference effects at high k's.

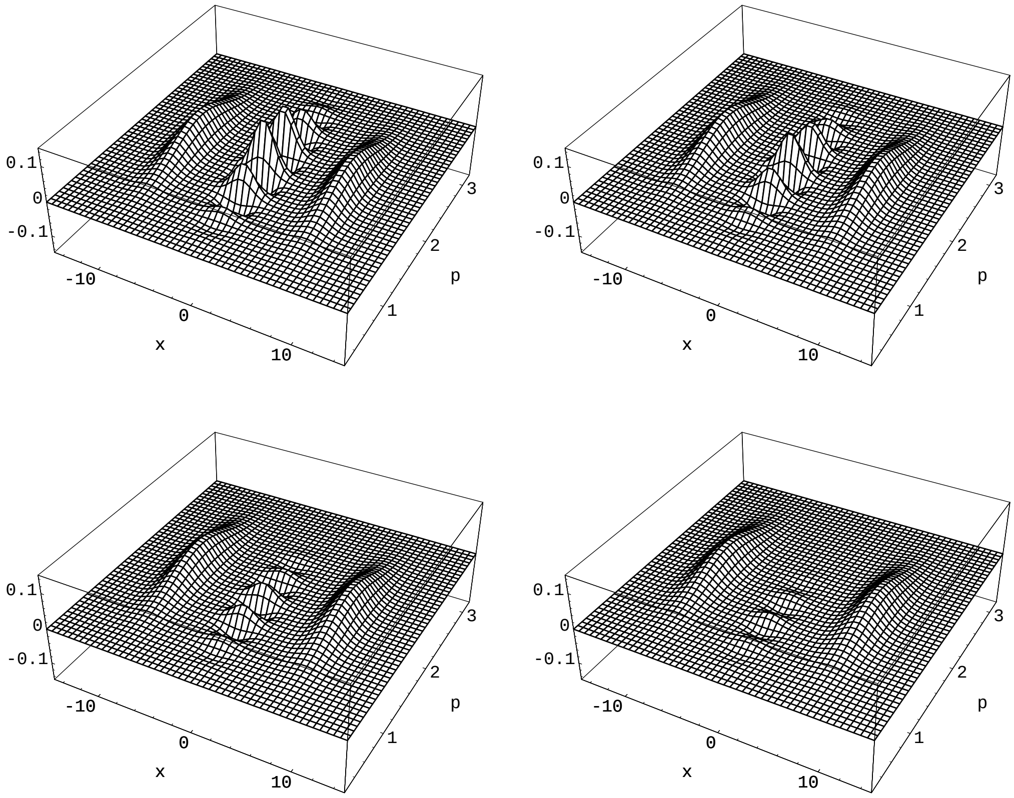

Fig. 2. Wigner function (11) for $\Delta_0 = 16.1 \cdot 10^{-10}$m, $k_0 = 1.7 \cdot 10^{10}\mathrm{m}^{-1}, \delta = 1.1 \cdot 10^{-10}$m; x is measured in units 10^{-10}m and $p = k$ in units $10^{10}\mathrm{m}^{-1}$. From above left to bottom right, $\sigma = 0, 0.5, 1.0, 1.5 \cdot 10^{-10}$m. Notice the asymmetry of the "wiggles" around k_0 and their fragility at high values of momentum. Observe also the slight "squashing" of the Gaussian components at high values of σ.

4 Decoherence parameter

One can give a quantitative estimate of the loss of quantum coherence by introducing a "decoherence parameter," in the same spirit of Refs. [5]. To this end, remember that the Wigner function can be expressed in terms of the density matrix ρ as

$$W(x,k) = \frac{1}{2\pi} \int d\xi \; e^{-ik\xi} \langle x + \xi/2 | \rho | x - \xi/2 \rangle, \tag{14}$$

and that $\mathrm{Tr}(\rho^2) = \mathrm{Tr}\rho = 1$ for a pure state, while $\mathrm{Tr}(\rho^2) < \mathrm{Tr}\rho = 1$ for a mixture. Define therefore the *decoherence parameter*

$$\varepsilon(\sigma) = 1 - \frac{\mathrm{Tr}(\rho^2)}{(\mathrm{Tr}\rho)^2} = 1 - \frac{2\pi \int dx \; dk \; W_{\mathrm{m}}(x,k)^2}{\left(\int dx \; dk \; W_{\mathrm{m}}(x,k)\right)^2}. \tag{15}$$

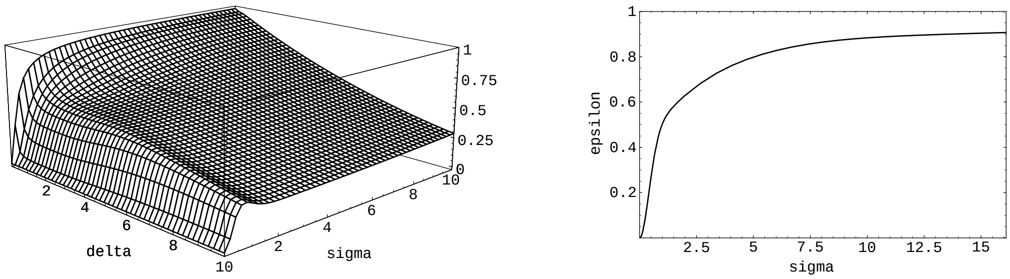

Fig. 3. Decoherence parameter. Left: ϵ as a function of δ and σ (both in units 10^{-10}m). Notice the peculiar behavior when $\delta > 3$ and $1 < \sigma < 2$. Right: ϵ vs σ (in 10^{-10}m) for $\delta = 1.1 \cdot 10^{-10}$m (experimental value in [3]).

This quantity is expected to vanish for $\sigma = 0$ (no fluctuation of the B-field and quantum coherence perfectly preserved) and to become unity when $\sigma \to \infty$ (large fluctuations of the B-field and quantum coherence completely lost). Figure 3 confirms these expectations, that can also be proven analitically from (11). In Ref. [3], $\delta = 1.1 \cdot 10^{-10}$m and σ is (presumably) very small, being the intensity of the B field controlled with high accuracy. It is remarkable that the decoherence parameter is not a monotonic function of the noise σ, when $\delta > 3 \cdot 10^{-10}$m and $1 \cdot 10^{-10}$m$< \sigma < 2 \cdot 10^{-10}$m. This may be due to our very definition (15) or to some physical effect we do not yet understand.

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