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QUANTUM PHASE PROPERTIES OF KERR COUPLERS*

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We use the concept of the phase space and the Husimi quasidistribution to study quantum phase properties of the optical fields propagating in Kerr couplers. Fourier coefficients of the phase distributions are introduced and utilized to examine their spatial development. The collapses and revivals of the mean photon number oscillations between the two waveguides are due to the bifurcation of the phase-difference probability distribution, which has a two-fold symmetry in the interval of collapse.

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1 Introduction

The aim of this paper is to study phase properties of the optical modes propagating in Kerr nonlinear coupler. The modes in the two waveguides, which form the coupler, are mutually linearly coupled and each mode is nonlinearly coupled to itself by means of Kerr effect. We also include the nonlinear cross-interaction. The analysis based on a classical wave-optical description reveals that the coupler exhibits several different regimes of operation in dependence on the values of linear and nonlinear coupling constants and input intensities [1]. A certain threshold value of the input intensity exists, the energy is periodically fully exchanged between the modes below the threshold, the intensities are asymptotically uniformly distributed at the threshold and finally, when the input intensity exceeds the threshold value, the nonlinear effect locks the energy in the mode, where it was initially injected, and the exchange of the energy decreases. The quantum Kerr coupler behaves quite differently from its classical counterpart below the threshold [2, 3]. An existence of collapses and revivals of the mean photon number

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oscillations was found in the sub-threshold region of operation. A possibility of generation of nonclassical states of light in Kerr coupler was discussed in [3].

In this paper we investigate the phase properties of the optical fields propagating in the coupler. The theory of quantum phase has attracted a strong attention during last years when several approaches to the quantum phase were proposed [4, 5] (for a review, see [6]). Quantum phase properties of elliptically polarized light propagating in a Kerr medium have been investigated within the framework of the Pegg-Barnett Hermitian phase formalism [7, 8]. In the present work we use the concept of a phase space approach [9], which has been recently applied to the investigation of phase properties of contradirectional nonlinear coupler operating by means of degenerate parametric down-conversion [10]. We introduce Fourier coefficients of the phase distributions and utilize them to investigate their spatial evolution.

2 Quantum dynamics of the coupler

Kerr coupler can be described with the help of an interaction momentum operator

$$\hat{G}_{\rm int} = \hbar\kappa(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}) + \hbar g\hat{a}^{\dagger 2}\hat{a}^2 + \hbar g\hat{b}^{\dagger 2}\hat{b}^2 + \hbar\tilde{g}\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}. \tag{1}$$

Here $\hat{a}$ ($\hat{a}^\dagger$) and $\hat{b}$ ($\hat{b}^\dagger$) are annihilation (creation) operators of the modes in the first and second waveguide in the interaction picture; κ is the linear coupling constant between the two modes, g and $\tilde{g}$ are nonlinear coupling constants proportional to the third-order susceptibility characterizing the strength of the self-action and cross-action processes, respectively. It is assumed that the propagation constants of both modes are the same. We will work in the Schrödinger picture and we find the evolution of the state vector $|\Psi(z)\rangle$, which is a solution of the Schrödinger equation,

$$-i\hbar\frac{\mathrm{d}}{\mathrm{d}z}|\Psi(z)\rangle = \hat{G}|\Psi(z)\rangle \tag{2}$$

and may be expressed in the basis of the Fock states $|n_1, n_2\rangle \equiv |n_1\rangle|n_2\rangle$,

$$|\Psi(z)\rangle = \sum_{n_1,n_2=0}^{\infty} c(n_1, n_2; z)|n_1, n_2\rangle, \quad c(n_1, n_2; z) = \langle n_1, n_2|\Psi(z)\rangle. \tag{3}$$

The solution (3) of (2) can be easily found numerically. This was described in detail in [2] and we point out only the most important facts. The Hilbert space of all states splits into a direct sum of invariant subspaces due to the conservation of the total number of photons (operator $\hat{n} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ commutes with the momentum operator), $\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n$. The dimension of each subspace $\mathcal{H}_n$ is $n+1$ and the basis can be chosen in the form $|j, n-j\rangle$, $j = 0, \ldots, n$. The Schrödinger equation is rewritten into matrix form for each subspace, the eigenvalues $\lambda_{n,j}$ and eigenvectors $|\Psi_{n,j}\rangle$ of finite matrices $\mathbf{G}_n$ (which form matrix representation of the momentum operator in each Hilbert space $\mathcal{H}_n$) are found and the initial conditions are applied to obtain the coefficients $c(m_1, m_2; z)$. If initial coherent states with complex amplitudes α, β are assumed, then

$$c(m_1, m_2; 0) = \frac{\alpha^{m_1}\beta^{m_2}}{\sqrt{m_1!\,m_2!}}\exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right). \tag{4}$$

Of course, we cannot sum over the whole Hilbert space, but we must restrict ourselves to the finite number of subspaces. We must choose $n_{\max}$ so that the actual Hilbert subspace is large enough to contain all important information about the process under discussion. The upper limit depends significantly on the total mean number of photons $\bar{n} = \langle \hat{n} \rangle$, for our calculations $\bar{n} = 4$ and $n_{\max} = 20$ has proved to be sufficient.

3 Phase space approach to the quantum phase

3.1 Phase distributions

The joint-phase probability distribution can be defined via radial integration of the Husimi quasidistribution [9]

$$P(\varphi_1,\, \varphi_2) = \int_0^\infty \int_0^\infty \Phi_{\mathcal{A}}(\alpha_1, \alpha_1^*; \alpha_2, \alpha_2^*)|\alpha_1||\alpha_2| \mathrm{d}|\alpha_1| \mathrm{d}|\alpha_2|, \tag{5}$$

where $\alpha_j = |\alpha_j| \exp(i\varphi_j),\ j = 1,\ 2$. The two-mode Husimi quasidistribution corresponding to the pure state $|\Psi\rangle$ is defined as

$$\Phi_{\mathcal{A}}(\alpha_1, \alpha_2) = \frac{1}{\pi^2} \left| \langle \alpha_1, \alpha_2 | \Psi \rangle \right|^2 , \tag{6}$$

where $|\alpha_1, \alpha_2\rangle$ is two-mode coherent state. Further we introduce the density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi|$ whose elements in the basis of the Fock states are

$$\rho(m_1, m_2; n_1, n_2) \equiv \langle m_1, m_2 | \hat{\rho} | n_1, n_2 \rangle = c(m_1, m_2) c^*(n_1, n_2). \tag{7}$$

Substitution of (6) into (5) and integration yield

$$\begin{aligned} P(\varphi_1,\, \varphi_2) \quad = \quad & \frac{1}{(2\pi)^2} \sum_{m_1, n_1 = 0}^{\infty} \sum_{m_2, n_2 = 0}^{\infty} \rho(m_1, m_2; n_1, n_2) \\ & \times \frac{\Gamma(\frac{m_1+n_1}{2} + 1)\Gamma(\frac{m_2+n_2}{2} + 1)}{\sqrt{m_1!\, n_1!\, m_2!\, n_2!}} e^{i(n_1-m_1)\varphi_1} e^{i(n_2-m_2)\varphi_2}. \end{aligned} \tag{8}$$

An integration over φ_1 or φ_2 leads to the single-mode phase distribution

$$P_j(\varphi) = \frac{1}{2\pi} \sum_{m,n=0}^{\infty} \frac{\rho_j(m,n)\Gamma(\frac{m+n}{2} + 1)}{\sqrt{m!\, n!}} e^{i(n-m)\varphi}, \quad j = 1, 2. \tag{9}$$

The density matrices of single modes $\hat{\rho}_j$ can be expressed as traces over the other mode, $\hat{\rho}_1 = \mathrm{Tr}_2 \hat{\rho}$ and $\hat{\rho}_2 = \mathrm{Tr}_1 \hat{\rho}$.

Probability distribution of the phase difference $\Delta\varphi = \varphi_1 - \varphi_2$ can be determined as an average of the Dirac function $\delta(\Delta\varphi - \varphi_1 + \varphi_2)$ over $P(\varphi_1, \varphi_2)$ yielding

$$\begin{aligned} P(\Delta\varphi) \quad = \quad & \frac{1}{2\pi} \sum_{m_1, n_1 = 0}^{\infty} \sum_{m_2 = \max(0, n_1 - m_1)}^{\infty} \rho(m_1, m_2; n_1, m_2 + m_1 - n_1) \\ & \times \frac{\Gamma(\frac{m_1+n_1}{2} + 1)\Gamma(\frac{2m_2+m_1-n_1}{2} + 1)}{\sqrt{m_1!\, n_1!\, m_2!\, (m_2 + m_1 - n_1)!}} e^{i(n_1-m_1)\Delta\varphi}. \end{aligned} \tag{10}$$

3.2 Fourier coefficients and phase dispersion

In order to study the spatial or temporal development of the single-mode distributions $P_j(\varphi)$ and the phase difference distribution $P(\Delta\varphi)$ it is useful to utilize the Fourier coefficients of the corresponding distributions, $F_j(k) = |\langle \exp(ik\varphi)\rangle|$ or $F(k) = |\langle \exp(ik\Delta\varphi)\rangle|$. For single-mode distributions we arrive at

$$F_j(k) = \left| \sum_{n=0}^{\infty} \frac{\rho_j(n+k,n)\Gamma(\frac{2n+k}{2}+1)}{\sqrt{n!\,(n+k)\,!}} \right|, \qquad j=1,2. \tag{11}$$

These coefficients provide us with information about the amplitude of k-fold modulation of the corresponding phase distribution and Husimi quasidistribution. There is also a close relation to the phase dispersion $\sigma^2 = 1 - |\langle e^{i\varphi}\rangle|^2$ which has been introduced in [11]. It follows that $\sigma^2 = 1 - F^2(1)$. We can generalize this relation and introduce higher-order dispersions $\sigma(k)$,

$$\sigma^2(k) \equiv 1 - |\langle \exp(ik\varphi)\rangle|^2 = 1 - F^2(k). \tag{12}$$

When dispersions increase, then the modulation of $P_j(\varphi)$ becomes less pronounced and phase uncertainty increases.

A closer inspection of the formulas (9) and (10) reveals that the expressions for the single mode phase distribution $P_j(\varphi)$ and the phase difference distribution $P(\Delta\varphi)$ are formally equivalent, if we define an effective density matrix for the latter case,

$$\begin{aligned} \tilde{\rho}(m,n) &= \sum_{s=0}^{\infty} \rho(m,s;n,s+m-n)\frac{\Gamma(\frac{2s+m-n}{2}+1)}{\sqrt{s!\,(s+m-n)!}}, && m \geq n; \\ \tilde{\rho}(m,n) &= \tilde{\rho}^*(n,m), && m < n. \end{aligned} \tag{13}$$

This effective density matrix may be inserted into (11), which yields the required Fourier coefficients of the phase-difference distribution.

The present formulas differ from those obtained using Pegg–Barnett formalism by numerical coefficients $\Gamma((m+n)/2+1)/\sqrt{m!\,n!}$. The distributions (8), (9) and (10) are generally broader than those which follow from Pegg–Barnett formalism.

4 Discussion

In this section we briefly present the most important results. A more detailed discussion will appear in [13]. We assume that initially the first mode is in the coherent state and the second mode is in the vacuum state. It is useful to remember the results obtained for the light propagation in Kerr media [7, 8]. The authors found out that the evolution can be periodic and that for $z = MZ/N$, where M, N are integers and $Z = \pi/g$ is a period, the field becomes a superposition of finite number of coherent states. In our case the linear coupling destroys the periodicity of the evolution. However, this periodicity can be restored when $\tilde{g} = 2g$, i.e. the self- and cross-phase modulations compensate each other. Then the energy is periodically exchanged between the two waveguides and the coupler behaves like a linear one. This is reflected in the phase-difference distribution which has the form corresponding to the coherent states in the both modes. Nevertheless, the single modes are not in coherent states. The single mode phase

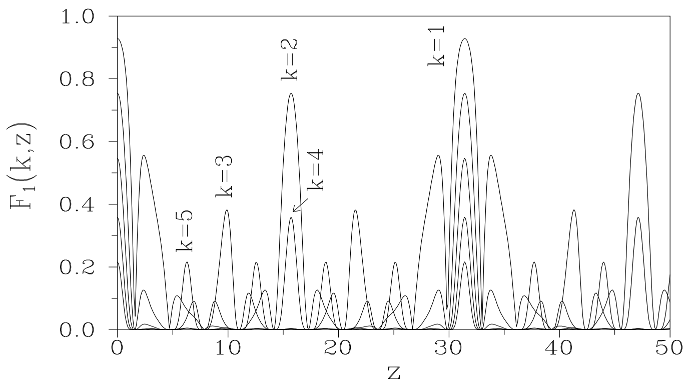

Fig. 1. Spatial evolution of the Fourier coefficients $F_1(k, z)$ of the first mode phase distribution for $\kappa = 1$, $g = 0.1$, $\tilde{g} = 0.2$, $\alpha = 2$ and $\beta = 0$.

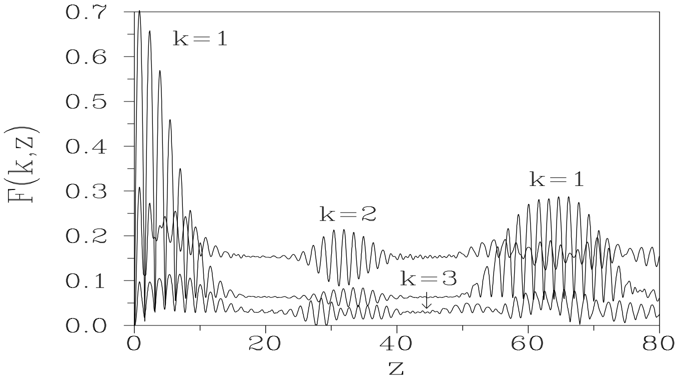

Fig. 2. Spatial evolution of the Fourier coefficients $F_1(k, z)$ of the phase-difference distribution for $\kappa = 1$, $g = 0.1$, $\tilde{g} = 0$, $\alpha = 2$ and $\beta = 0$.

distribution and Husimi quasidistribution exhibit N-fold modulation for $z = MZ/N$, which is reflected in peak values of the corresponding Fourier coefficients, see Fig. 1. Note that peaks for $k = 4$ are missing, because the first mode is in vacuum state for $z = Z/4$. More generally the evolution resulting from nonlinear phase modulation is superimposed with fast oscillations induced by κ.

If the input intensity is lower than a threshold value and $\tilde{g} \neq 2g$, then an interval of collapse of oscillations between the two waveguides arises [2, 3]. This effect is strongest for $\tilde{g} = 0$. We can see a typical spatial evolution of the Fourier coefficients $F(k)$ of the phase–difference distribution in this case in Fig. 2. In the interval of collapse, $z \in (10, 50)$, the inequality $F(2) > F(1)$ holds. The distribution $P(\Delta\varphi)$ has a two-fold symmetry in this interval, the two peaks are placed at $\Delta\varphi = \mp\pi/2$. These two peaks are responsible for simultaneous flow of the energy from the first to the second waveguide and vice versa. Notice that a similar bifurcation of a single mode phase distribution has been obtained for highly squeezed states [10, 12]. Finally, the revival takes place and the energy exchange is re-established. However, it is weaker and only a part of the energy is periodically exchanged.

5 Conclusions

We have investigated quantum phase properties of optical fields propagating in Kerr couplers. We have utilized the Fourier coefficients to investigate spatial evolution of the phase distributions. Single-mode phase distributions can exhibit k-fold symmetry induced by nonlinear phase modulation. Bifurcation of the phase-difference distribution appears in the interval of collapse of the energy oscillations between the waveguides.

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