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RECONSTRUCTION OF QUANTUM STATES WITH BINARY DETECTORS*

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The scheme is proposed to perform the reconstruction of a multi-mode quantum state of light with help of non-ideal detectors able to test only presence or absence of photons.

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A binary detector able to test only presence or absence of photons is the simplest device which could provide an information about the quantum state of light. Examples of such detectors are the avalanche photodiodes operating in the Geiger mode. They are not capable of resolving the number of photons that triggered the output signal. Such detectors are the most efficient ones available now for the single-photon level light. Recently it was shown [1-3] that it is possible to reconstruct the state of a single light mode using such binary detectors. It was also shown [4] that binary detectors could be used for demonstration of non-locality of two-mode correlated state of light. The aim of current work is to demonstrate how the density matrix of an arbitrary multi-mode state of light in the Fock-state basis can be inferred from the data provided by the set of binary detectors. It should be noted, that in difference with well known and extensively developed method of optical homodyne tomography (for a review, see [5]), this reconstruction scheme does not impose a lower limit on the detector efficiency.

The basic scheme of the measurement set-up is as follows (Fig. 1): let the $N+1$-mode state be an input of the $2(N+1)$-port linear device, which performs a unitary transformation $U^{(N+1)}$ over this state and produces $N+1$ output modes. Then each of these modes is measured by the corresponding binary detector.

It's worth to mention that the similar scheme (named "photon chopping") using multi-port device were proposed by Paul and co-workers in the work [6] for the reconstruction of single-mode state. $2(N+1)$-port linear device of our set-up performs $SU(N+1)$ rotation over the annihilation a_i and creation a_i^+ operators of the input modes. The annihilation b_i and creation b_i^+ operators of the output modes could be expressed as

$$b_i^{(\dagger)} = U^{(N+1)\dagger} a_i^{(\dagger)} U^{(N+1)} = \sum_{j=0}^{N} U_{ij}^{(*)} a_j^{(\dagger)}. \tag{1}$$

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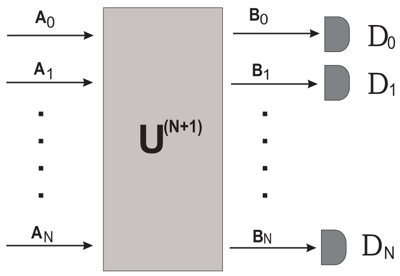

Fig. 1. Schematic outline of the measurement set-up for the reconstruction of multi-mode state.

Such transformation could be realized using beam-splitters and phase-shifters. Currently we restrict ourselves to measurement of the probability of simultaneous zero counts on all detectors. It is given by the Mandel's formulae

$$p_0 = \langle : \exp\left(-\sum_{i=0}^{N} \eta_i b_i^+ b_i\right) : \rangle,$$

where η_i denotes the quantum efficiency of the ith detector and :: denotes the normal ordering. If the density matrix of the signal state could be represented in the Fock-state basis as

$$\rho = \sum_{M,L=0}^{\infty} \sum_{\sum i_j = M, \sum l_j = L} \rho_{i_0 \ldots i_N}^{l_0 \ldots l_N} |i_0\rangle \ldots |i_N\rangle\langle l_N| \ldots \langle l_0|,$$

the probability p_0 could be expressed in the following way

$$p_0 = \sum_{M=0}^{\infty} \sum_{\sum i_j, k_j, l_j = M} \prod_{j=0}^{N} (1-\eta_j)^{i_j} S_{i_0 \ldots i_N}^{k_0 \ldots k_N} \left(S_{i_0 \ldots i_N}^{l_0 \ldots l_N}\right)^* \rho_{k_0 \ldots k_N}^{l_0 \ldots l_N}, \tag{2}$$

where the coefficients $S_{i_0 \ldots i_N}^{k_0 \ldots k_N}$ are defined by the transformation (1)

$$S_{i_0 \ldots i_N}^{k_0 \ldots k_N} = \langle i_N| \ldots \langle i_0| U^{(N+1)\dagger} |k_0\rangle \ldots |k_N\rangle$$

The equation (2) is the basic tool of the reconstruction procedure. Essentially, it stems from the fact that for any finite-sized density matrix of signal state it is possible to measure p_0 for different parameters of $2(N+1)$-port and efficiencies of the detectors and obtain the system of linear algebraic equations for the coefficients $\rho_{i_0 \ldots i_N}^{l_0 \ldots l_N}$. Immediately, three important conclusions follow from the formulae (2). First, it is obvious, that such reconstruction scheme would work only if the detectors are non-ideal. One should have a non-zero probability to obtain no photocounts when state with arbitrary high number of photons impinges on the detector. Second, if no information available about the signal density matrix apart from its finiteness, the scheme does not allow to make the complete reconstruction. Namely, it allows to find such subset of the elements $\rho_{i_0 \ldots i_N}^{l_0 \ldots l_N}$ that $\sum i_j = \sum l_j$. This subset includes, for example, the diagonal elements of the density matrix. Third, if the density matrix of the signal state may be represented as a direct product of known single-mode density matrix σ and unknown N-mode density matrix γ, the expression (2) could be re-written as

$$p_0 = \sum_{M=0} \sum_{\sum i_j = M} \sum_{\sum k_j, l_j \le M} \prod_{j=0}^{N} (1-\eta_j)^{i_j} S_{i_0 \ldots i_N}^{k_0 \ldots k_{N-1}, M - \sum k_j} \times \left(S_{i_0 \ldots i_N}^{l_0 \ldots l_{N-1}, M - \sum l_j} \right)^* \gamma_{k_0 \ldots k_{N-1}}^{l_0 \ldots l_{N-1}} \sigma_{M - \sum l_j}^{M - \sum k_j},$$

where sums of k_j and l_j is taken from $j = 0$ to $j = N - 1$. As seen from this equation, it is sufficient to use *one* probe state only for the reconstruction of N-mode signal state. Of course, to make the reconstruction possible one should be able to choose the parameters of the $2(N+1)$-port, the detector efficiencies and the probe state density matrix in such way that the obtained system of equations is non-degenerate. For example (as it was shown in [1,2]), for the reconstruction of the single-mode state the 4-port is simply a beam-splitter, and it is possible to use coherent states for the probe.

Now we demonstrate how the described scheme works using a simple example of reconstruction of two-mode completely unpolarized light. The density matrix of completely unpolarized light has the form [7,8]

$$\rho = \sum_{M=0} R_M \rho_M, \tag{3}$$

where the operators ρ_M describe states of precisely M photons with completely undefined polarization

$$\rho_M = \frac{1}{M+1} \sum_{j=0}^{M} |j_0\rangle \, |(M-j)_1\rangle \langle (M-j)_1| \, \langle j_0| \, .$$

of two orthogonally or circularly polarized modes a_0 and a_1 .The measurement set-up for the reconstruction of the state (3) is actually the direct detection scheme (Fig. 2). As it could be easily seen, for the detector with the efficiency η_i the probability to have no photocounts is given by

$$p_0^{(i)} = \langle : \exp\left(-\eta_i \left(a_0^+ a_0 + a_1^+ a_1\right)\right) : \rangle = \sum_{M=0} (1-\eta_i)^M R_M. \tag{4}$$

From the Eq. (4) it follows that the set of detectors with different efficiencies is enough for the reconstruction. Indeed, let the density matrix (3) be finite with the maximal number of photons $M_{max} = N$. In this case measuring p_0 for N different efficiencies of detectors we will obtain the non-degenerate system of linear equations which could be solved analytically with help of Lagrange polynomial interpolation [3,9].

Generally, our reconstruction procedure includes three basic components. First, it is the solution of the deterministic reconstruction problem described by the Eq. (4) [or, in general, by the Eq. (2)] even if it is overcomplete, i.e. more different $p_0^{(i)}$ are measured than the suggested maximal number of photons N. Second, an adequate tool should be provided for treatment of

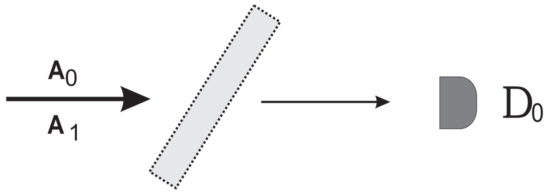

Fig. 2. Schematic outline of the measurement set-up for the reconstruction of two-mode completely unpolarized state.

the measurement noise. An influence of noise generally leads to incompatibility of the over-complete system of reconstruction equations, and makes necessary to implement a statistical estimation procedure [10,11]. Such procedure is also necessary to guarantee that the result of the reconstruction is indeed the density matrix, i.e. that the reconstructed matrix is positive definite. Third, it would be useful if we are able to check whether our initial suggestion about the size of the signal density matrix is correct. In other words, in our example it should be possible to perform an estimation for the sequence of suggested maximal numbers of photons N_1, N_2 and so on, and then to check how results of estimation converge with increasing of N.

In the works [1-3,10,11] the maximum likelihood estimation procedure was suggested for the purposes listed above. However, in our example of the reconstruction of completely unpolarized light a simpler procedure could be used for the same purpose, namely, the least square method with linear constraints [12]. Then it is possible to impose the positiveness of the result as a bound. It should be noted that, in general, the system of Eqs. (4) is ill-conditioned. The problem of finding a stable, numerically robust solution is far from trivial and requires a regularization procedure. In particular, in our consideration we used the stochastic regularization procedure [13].

For illustration we take the completely unpolarized thermal state described by the coefficients

$$R_n = \frac{N_{(s)}^n}{\left(1 + N_{(s)}\right)^{n+1}},$$

where $N_{(s)}$ denotes the average number of photons. The reconstruction procedure is as follows: at first, p_0 are measured for the number of different detector efficiencies. This number should be no less than the maximal truncation number. Of course, for practical purposes it is more convenient to use a variable absorber in from of just a single detector instead of a set of different detectors. A convenient model of such absorber is a beam-splitter. By varying its transmittance a set of detectors with different efficiencies can be simulated. Then the reconstruction is done for different suggested truncation numbers using obtained balk of data and the convergence of reconstructed elements is checked. In Fig. 3 an example of such procedure is shown for the completely unpolarized thermal state with $N_{(s)} = 0.5$. Here for plots (b,c) an infinite number of measurements was taken, so the statistical error is absent. Errors are caused by choosing the truncation number only. It should be noted that the convergence of the elements R_n is different for different n. For example, R_0 is well estimated even for the truncation number $N = 3$. In Fig. 3(d-f) it is shown using Monte-Carlo simulation how the statistical error propagates for the fixed truncation number. It could be seen that even the modest number of measurements

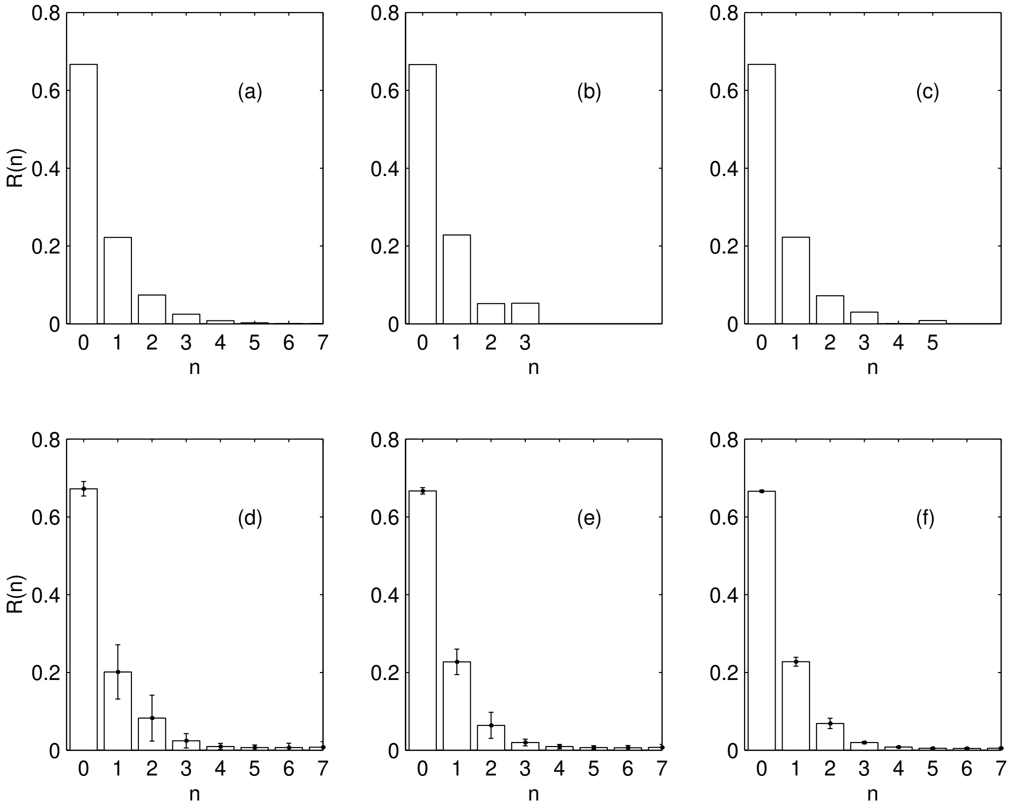

Fig. 3. Reconstruction of two-mode completely unpolarized thermal state with the average number of photons $N_{(s)} = 0.5$. On the plot (a) true coefficients R_n are plotted. On the plots (b,c) results of reconstruction are plotted for truncation numbers $N = 3$ and 5. For these plots an infinite number of measurements was taken, so the errors are caused by the truncation only. On the plots (d-f) the propagation of statistical error is illustrated for the truncation number $N = 7$ and for 25, 100 and 625 measurements for each value of detector efficiency. 50 different efficiencies were taken for all plots. They are homogeneously distributed in the interval $[0.1, 0.9]$.

provides good accuracy of the reconstruction. In particular, for the truncation number $N = 7$ only hundred measurements for each detector efficiency are enough [Fig. 3(e)]. It should be noted that again the convergence of the elements is different for different n . For higher n it is necessary to perform more measurements to achieve a sufficient accuracy.

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