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TESTING OF OPERATIONAL PHASE CONCEPTS*

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Various phase concepts may be treated as special cases of the maximum likelihood estimation. For example, the discrete the operational phase of Noh, Fougères and Mandel is obtained for continuous Gaussian signals with phase modulated mean. Although the Gaussian estimation gives a satisfactory approximation for fitting the phase distribution of almost any state the optimal phase estimation offers in certain cases a measurably better performance. This has been demonstrated in a neutron–optical experiment.

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1 Introduction

Phase measurements belong to standard detection techniques revealing the wave property of the signal. There are several concepts for the description of phase in quantum theory at present. For an up to date overview see [1, 2]. Some of them emphasize the theoretical aspects, others the experimental ones. The operational approach formulated by Noh, Fougères and Mandel (NFM) [3] is motivated by the correspondence principle in classical wave theory. The interference pattern is adopted for the scheme where the $\sin$ and $\cos$ function of the phase shift are measured simultaneously in the eight-port homodyne detection. The NFM scheme is plausible whenever the signal behaves like a classical wave since besides the principle of correspondence, no other assumption has been used. As a particular result, the NFM scheme provides the optimum result, provided that the statistics of the signals is represented by Gaussian statistics with a phase sensitive mean and a phase insensitive noise. Since realistic signals in the quantum world do not

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meet these rather restrictive criteria, phase prediction based on them is not optimum in general. The difference between Gaussian estimation and optimum treatments can be experimentally registered in an interferometer with discrete Poissonian signal.

2 Statistical formulation of operational phase concepts

In this section the operational phase concept will be naturally embedded in the general scheme of quantum estimation theory [4, 5]. A similar approach has already been used in [6]. Assume an ideal device with four output channels enumerated by indices 3,4,5,6, where the actual values of intensities are registered in each run. The values fluctuate in accordance with the statistics of continuous Gaussian signals. The mean intensities are modulated by a phase parameter θ as

$$\bar{I}_{3,4} = \frac{I}{2}(1 \pm V\cos\theta), \qquad \bar{I}_{5,6} = \frac{I}{2}(1 \pm V\sin\theta), \tag{1}$$

where I and V are total input intensity and visibility of the interference fringes, respectively. The energy is split symmetrically between all the output ports. This device represents nothing else than a classical wave picture of the original eight-port homodyne detection scheme. Equivalently, it also corresponds to a Mach-Zehnder interferometer, when the measurement is done for an unknown phase shift together with a zero and a $\pi/2$ auxiliary phase shifter. In this case, the data are not obtained simultaneously, but should be collected during repeated experiments. Provided that a particular combination of outputs I_3, I_4, I_5, I_6 has been registered, the phase shift should be inferred. In accordance with the MaxLik approach [4], the sought-after phase shift is given by the value that maximizes the likelihood function. The likelihood function corresponding to the detection of given data reads

$$\mathcal{L}(\theta, V, I) = \frac{1}{\sigma^4 4\pi^2}\exp\left\{\frac{1}{2\sigma^2}\left(-[I_3-\bar{I}_3]^2-[I_4-\bar{I}_4]^2-[I_5-\bar{I}_5]^2-[I_6-\bar{I}_6]^2\right)\right\}. \tag{2}$$

Here the variation σ represents the phase insensitive noise of each channel. The sampling of intensities may serve for an estimation of phase shift, the average number of particles, and the visibility simultaneously. Introducing the complex parameter $R = \sqrt{(I_3-I_4)^2+(I_5-I_6)^2}$, the likelihood function (2) is maximized by the choice of parameters

$$e^{i\theta} = \frac{1}{R}[I_3 - I_4 + i(I_5 - I_6)], \quad V = \min\left(\frac{2R}{\sum_{i=3}^{6} I_i}, 1\right), \quad I = \frac{1}{2}\sum_{i=3}^{6} I_i. \tag{3}$$

Hence the operational phase concept of Noh, Fougères and Mandel is nothing but the MaxLik estimation for waves represented by continuous Gaussian signal with phase–independent and symmetrical noises. These rather strong assumptions are associated with the behavior of waves in classical field theory.

3 Phase estimation in neutron interferometry

Assume the following modification of the scheme developed above. The output ports of an interferometer are considered to be nonsymmetrical. The measurement is done with many auxiliary phase shifters. Neutron beams inside the perfect crystal neutron interferometer will be described at first as a classical wave. The signals I_j^o, I_j^h detected at the two output ports o,h are regarded as stochastic Gaussian intensities with phase sensitive means

$$\bar{I}_j^o = I^o + I^V \cos(\theta + \Delta_j), \qquad \bar{I}_j^h = I^h - I^V \cos(\theta + \Delta_j), \tag{4}$$

where θ is the true phase shift between the two branches of the interferometer, I^o and I^h are the mean intensities of the two interference fringes, I^V stands for the modulation amplitude of the interference fringes (unnormalized visibility), and Δ_j are the values of the auxiliary shifts $\Delta_j = 2\pi j/N$, $j = 0, \ldots, N-1$, $N \in \mathcal{N}$. The Gaussian statistics of the detected signal yields its likelihood function in the form

$$\mathcal{L}(\theta) \propto \exp\left\{ -\frac{1}{2\sigma^2} \sum\nolimits_{j=0}^{N-1} \left[(I_j^o - \bar{I}_j^o)^2 + (I_j^h - \bar{I}_j^h)^2 \right] \right\}. \tag{5}$$

Introducing the complex parameter $R = \sum_j (I_j^o - I_j^h) \exp(-i\Delta_j)$, the parameters maximizing the likelihood function read

$$e^{i\theta} = R/|R|, \quad I^o = \frac{1}{N} \sum\nolimits_j I_j^o, \quad I^h = \frac{1}{N} \sum\nolimits_j I_j^h, \quad I^V = \min\{|R|/N, I^o, I^h\}. \tag{6}$$

The relation on estimated visibility follows from the condition of semidefiniteness of the amplitude $I^V \leq \min\{I^o, I^h\}$. Expression (6) represents a generalization of the NFM formula (3), which may be recovered provided that the measurement is done at the two positions $\Delta_0 = 0$, $\Delta_1 = \pi/2$ only. Sometimes it happens that $R = 0$ for recorded data. In this case the data are phase insensitive and the posterior phase distribution is homogeneous. Notice that the Gaussian phase formula (6 is well known in optics as phase estimator of discrete Fourier transformation (DFT) [7].

The generalization of NFM scheme given by phase estimation (6) is not optimal provided that the detected signal is Poissonian. In this case the likelihood function reads

$$\mathcal{L}(\theta) \propto \prod\nolimits_{j=0}^{N-1} \prod\nolimits_{\alpha=o,h} \left(\bar{I}_j^\alpha \right)^{n_j^\alpha} e^{-\bar{I}_j^\alpha}, \tag{7}$$

where the mean output intensities $\bar{I}_j^\alpha$ are given by Eq. (4) and n_j^α represent the number of detected neutrons. Unfortunately, the explicit relations analogous to (6) cannot be found analytically. The analysis must therefore be carried out numerically. However, various limiting and asymptotic cases can be discussed analytically as shown in [7]. DFT estimation is the best estimation available provided a large number of particles is detected. No difference between Gaussian and Poissonian cases can be therefore expected in the regime of high intensities $I_o, I_h \gg 1$. In the opposite case of low output intensities $I_o, I_h \ll 1$, the most frequent samples are "no detection at all" and "one neutron detected" in some beam at some value of the auxiliary shift. Direct substitution of these samples in both the Poissonian and the Gaussian likelihood functions tends to the prediction with undefined phase with $I^V = 0$. Similarly to the previous case of strong field, where both methods are equally good, in the low field limit both methods are equally bad. Similarly in the limit of low visibility $I^V \ll \min\{I_o, I_h\}$, the results obtained with the use of the true Poissonian statistics are virtually identical to those yielded by the DFT [7]. No difference between the wave and quantum approach can be observed here. The difference between the predictions for the quantum and wave phase estimation in a realistic experiment should become significant for visibility close to unity and average number of detected particle of order one.

4 Experiment

Our experiments were performed at the neutron interferometry setup at the 250 kW TRIGA reactor in Vienna. Thermal neutrons emitted from reactor behave as independent (Poissonian) particles [8]. Therefore, the neutron beam can serve us as an ideal signal for testing of our theory. The input beam is split by the perfect crystal interferometer into two partially coherent beams. One of the beams passes a phase plate which introduces an unknown phase shift which has to be estimated from the experimental data. Then both beams are passing an auxiliary phase shifter which modulates the output intensities at the detectors o and h. In our experiment eight equidistant positions of the phase shifter were used for generation of the intensity modulation.

To compare the efficiency of the NFM phase predictions with the optimum Poissonian MaxLik estimation the following procedure has been chosen. Each sample of data consisting of the number of neutrons counted in beams o, h in eight positions $\Delta_0, \ldots, \Delta_7$, was processed using NFM formula (6) resulting in phase prediction θ_{NFM}. The relative frequency $f_g(\Delta\theta)$ characterizes how many times the estimated phase θ_{NFM} falls into the chosen phase window $\Delta\theta$ (confidence interval) around the true phase shift. The same procedure was repeated for phase prediction based on numerical maximization of the Poissonian likelihood function (7) yielding the relative frequency of "hits" $f_p(\Delta\theta)$. The quantity

$$\Delta E \quad = \quad f_p(\Delta\theta) - f_g(\Delta\theta) \tag{8}$$

represents the difference in efficiency of the quantum and wave phase estimations for the given phase window $\Delta\theta$. If this quantity is found to be positive, it means that the MaxLik estimation is better than its Gaussian counterpart.

Analysis of the experimental data is summarized in Fig. 1. The difference ΔE was calculated using 690 experimental samples measured in experiment with average beam intensities I^o=2.2 neutrons, I^h=6.3 neutrons, and visibility normalized with respect to the o beam being about 47%. This represents a critical situation, because even though there are less than 10 counts in each experimental run, one is nonetheless trying to get an estimation of the value of the phase shift. The experimental values of ΔE are depicted in Fig. 1 by full squares. For comparison, a theoretical prediction corresponding to the same values of parameters as in the real experiment were simulated in the Monte Carlo experiment using 40 000 samples. Open circles in Fig. 1 show the corresponding mean values of the difference. However, since the experimental data are limited due to the experimental conditions and available time to the relatively small number of 690 samples, the real data are fluctuating around the mean values. Statistical significance of the experimental results is demonstrated again using Monte Carlo simulations. Another 20 simulations have been done, each of them with 690 samples. The variance of the ensemble $\{\Delta E_j\}$ is shown in Fig. 1 as "error bars" for each phase window.

A significant difference between the effectiveness of classical and optimum treatments is apparent in Fig. 1. The optimum estimation provides an improvement in fitting of the phase

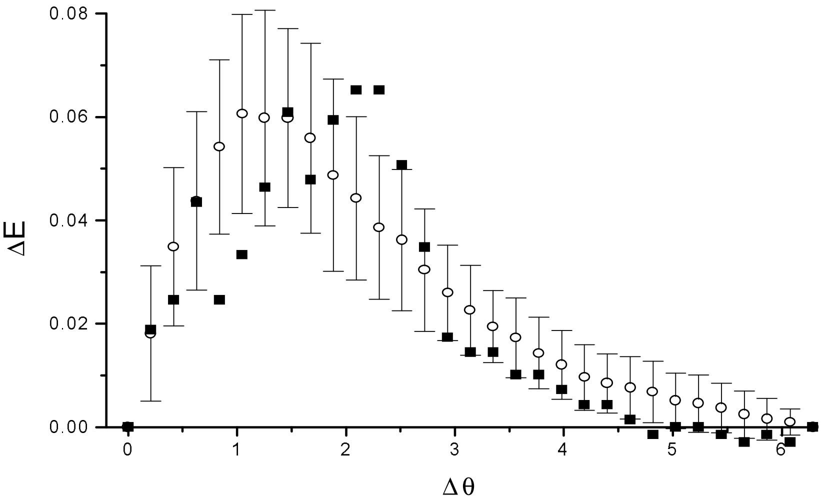

Fig. 1. Experimentally obtained ΔE denoted by symbols (■) are compared with theoretical prediction denoted by corresponding mean values (○) and error bars for 690 samples. $I^o = 2.21$ neutrons, $I^h = 6.33$ neutrons, $I^V = 1.03$ neutrons and $\theta = 4.83$ rad.

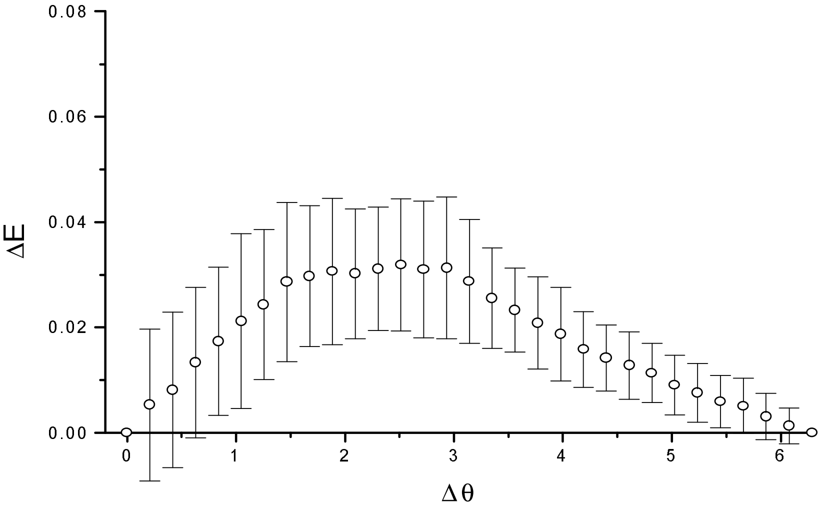

Fig. 2. Figure shows the difference between quantum and semiclassical phase estimation for an imperfect interferometric setup. The variance of the Gaussian phase fluctuations is equal to $\pi/5$. Other parameters are the same as in Fig. 1.

shift and the difference is beyond the estimation error, approximately 2.5 standard deviations in the optimum case. Obviously, no better performance of the MaxLik method can be expected for large values of the phase window $\Delta\theta$ (any sensible statistical method would yield rather reasonable results). Likewise, no real improvement over the Gaussian estimate can be expected when $\Delta\theta$ is close to zero, because too few data would then be accepted. The existence of a "best choice" for the phase window is therefore in itself an interesting feature of the method we propose. However, notice that the generalized NFM scheme (or equivalently DFT phase estimator) fits the phase shift quite well. The most pronounced difference is about 6% in the window of width about 1.256 rad. The observed difference is therefore not large, yet statistically significant.

Obviously, noise of the experimental setup will reduce the observed effect. It is therefore important to check robustness of our method with respect to various imperfections. For example, vibrations or temperature changes may introduce fluctuations of the path difference inside the interferometer. This can be described by allowing the true phase to fluctuate around its mean value according to some probability distribution. Fortunately, it turns out that stability of the interferometer is not critical for observation the difference between predictions of the semiclassical and quantum theories. In the presence of fluctuations the distribution of MaxLik phase estimates is effectively given by the convolution of the noise-free intrinsical phase distribution and the probability distribution of the phase fluctuations. Provided that the phase fluctuations are small in comparison to the intrinsical phase uncertainty, the observed ΔE cannot be influenced by the fluctuations too much. Since we are concerned mainly with the weak signals with rather uncertain phase, relatively large amount of noise is allowed without spoiling the measurement. This is illustrated with the help of computer simulation in Fig. 2 for the experimental setup described above. In spite of the fact that the substantial Gaussian fluctuations ($\sigma^2 = \pi/5$) are present, the difference between the semiclassical and quantum phase predictions is still observable. Thus the sensitivity to fluctuations provides another argument for importance of the intermediate regime of a few particles for experimental testing of various aspects of operational phase concepts.

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