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PHASE DISTRIBUTION OF THE MICROMASER FIELD WITH INJECTED ATOMIC COHERENCE*

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We present the solution for the phase distribution of the steady state micromaser field for the case with injected atomic coherence in the semiclassical approximation.

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1 Introduction

The micromaser pumped by atoms in their upper state is a device which has received a great deal of attention. The steady field it produces has no mean phase, because any value of the phase is equally likely. If the atoms are injected in a coherent superposition state, however, the atomic dipoles do have a phase, and because they drive the field, the field has a phase as well. Here we would like to investigate the phase distribution of the field which is produced by atoms in a coherent superposition state.

The micromaser with injected atomic coherence has been investigated by several authors, but only in restricted parameter regimes, for example, keeping terms in the master equation only up to 4th order in the atom-field coupling constant [1–3]. This is a much more complicated problem than the case in which the atoms are completely inverted. In that case, the diagonal elements of the density matrix couple only to each other, but when there is atomic coherence this is no longer the case. The entire density matrix must be treated at once.

Here we employ techniques developed by Gea-Banacloche to the problem [4]. These rely on the very simple time evolution of particular atomic states. We are able to use them to find an approximate expression for the phase distribution of the field which contains the mean photon number as a parameter. This must be found by other means, but once it is, the phase distribution is determined. Our results are compared with phase distributions obtained from numerical simulations, and the agreement is very good.

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2 Micromaser

We shall begin describing briefly micromaser [5]. A micromaser consists of a beam of two-level atoms passing through a high-Q microwave cavity with electromagnetic field. It is assumed that the atomic beam is of low density, i.e. at any given time there is at most one atom present in the cavity. We also assume that the microcavity supports only one mode of the field which is in resonance with the atoms at the frequency ω. The atom-field interaction is governed by the Jaynes-Cummings model and when the cavity is empty then the field decays into the environment which is in our case at zero temperature. If the probability of the atoms being at the upper state is high when they enter the microcavity then the micromaser can maintain non-zero steady state field.

The atoms have states $|a\rangle$, with energy ω (we are using units with $\hbar = 1$), and $|b\rangle$, with energy 0. The Hamiltonian describing the atom-field system is [6]

$$H = \omega a^\dagger a + \frac{1}{2}\omega(\sigma_3 + I) + g(a^\dagger \sigma^- + a\sigma^+). \tag{1}$$

If the atom is initially in the coherent state

$$|\Psi_{\rm at}\rangle = \alpha|a\rangle + \beta|b\rangle, \tag{2}$$

and the field is initially in the state

$$|f\rangle = \sum_{n=0}^{\infty} d_n|n\rangle, \tag{3}$$

then after a time τ the state of the combined system will be

$$\begin{aligned} |f\rangle \otimes (\alpha|a\rangle + \beta|b\rangle) \quad &\to \quad \sum_{n=0}^{\infty} d_n(\alpha c_{n+1}|n\rangle - \mathrm{i}\beta s_n|n-1\rangle)|a\rangle \\ &+ \quad \sum_{n=0}^{\infty} d_n(\beta c_n|n\rangle - \mathrm{i}\alpha s_{n+1}|n+1\rangle)|b\rangle, \end{aligned} \tag{4}$$

where $s_n = \sin(g\tau\sqrt{n})$ and $c_n = \sin(g\tau\sqrt{n})$ and g being the atom-field interaction constant. Eq. (4) defines the time evolution of the field density matrix $\hat{\rho} = \mathrm{Tr}_{\rm at}\hat{\rho}_{system}$ that can be written in the number-state representation as

$$\begin{aligned} \rho_{nm} \quad \to \quad & \rho_{nm}(|\alpha|^2 c_{n+1}c_{m+1} + |\beta|^2 c_n c_m) \\ + \quad & \rho_{n-1,m-1}|\alpha|^2 s_n s_m \\ + \quad & \rho_{n+1,m+1}|\beta|^2 s_{n+1}s_{m+1} \\ + \quad & \mathrm{i}\alpha\beta^*(c_{n+1}s_{m+1}\rho_{n,m+1} - s_n c_m \rho_{n-1,m}) \\ + \quad & \mathrm{i}\alpha^*\beta(c_n s_m \rho_{n,m-1} - s_{n+1}c_{m+1}\rho_{n+1,m}). \end{aligned} \tag{5}$$

The decay of the micromaser field for the cavity at zero temperature is described by the master equation [7]

$$\frac{d\hat{\rho}}{dt} = -\frac{1}{2}\gamma(a^\dagger a\hat{\rho} + \hat{\rho}a^\dagger a - 2a\hat{\rho}a^\dagger), \tag{6}$$

which has the solution in the number-state representation

$$\rho_{mn}(t) = \mathrm{e}^{-\gamma t(m+n)/2} \sum_{l=0}^{\infty} \left(\frac{(m+l)!}{m!} \frac{(n+l)!}{n!} \right)^{1/2} \frac{(1-\mathrm{e}^{-\gamma t})^l}{l!} \rho_{m+l,n+l}(0), \tag{7}$$

where γ is the loss coefficient. Eqs. (5,7) will be used in the numerical simulations.

3 Phase Distribution

Knowing the field density matrix at given time the phase distribution of the field can be calculated. The probability that the phase is within the interval $\lambda, \lambda + d\lambda$ is $\langle \lambda | \hat{\rho} | \lambda \rangle d\lambda$, where $|\lambda\rangle$ is the quantum-mechanical phase state

$$|\lambda\rangle \equiv \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\lambda} |n\rangle. \tag{8}$$

The core of our approach to determining the phase distribution lies in using the special orthogonal atomic states

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}} (\mathrm{e}^{-\mathrm{i}\phi} |a\rangle \pm |b\rangle), \tag{9}$$

as the atomic basis. $|\pm\rangle$ are actually the eigenstates of the semiclassical interaction Hamiltonian which is obtained when replacing the operators a $(a^\dagger)$ in the interaction term of Eq. (1) by the c-number field amplitude v (v^*), $v = |v|\mathrm{e}^{-\mathrm{i}\phi}$,

$$H_{\text{int,sc}} = g(\sigma^- v^* + \sigma^+ v). \tag{10}$$

These states were intensively studied within the Jaynes-Cummings model by J. Gea-Banacloche [4] who discovered relatively simple formula which describes their time evolution in the limit of very large photon number $\bar{n}$. On the time scale which is much smaller than the revival time $t_{\text{r}} = 2\pi\sqrt{\bar{n}}/g$, the micromaser field is very similar to coherent state and the system initially prepared in one of the states $|\pm\rangle|v\rangle$ evolves according

$$|\pm\rangle|v\rangle \rightarrow \mathrm{e}^{\mp\mathrm{i}\frac{gt\sqrt{\bar{n}}}{2}} |\pm\rangle |v\mathrm{e}^{\mp\mathrm{i}\frac{gt}{2\sqrt{\bar{n}}}}\rangle, \tag{11}$$

where $|v\rangle$ is the initial coherent field with complex amplitude $v = \sqrt{\bar{n}}\mathrm{e}^{-\mathrm{i}\phi}$. The time evolution given by Eq. (11) is good up to the order of $1/\sqrt{\bar{n}}$ having $\bar{n} = |v|^2$.

An arbitrary atomic state given by Eq. (2) can alternatively be expressed as

$$|\Psi\rangle = \mathrm{e}^{-\mathrm{i}u} \sin\theta |a\rangle + \cos\theta |b\rangle, \tag{12}$$

and then in the basis of $|\pm\rangle$ states as

$$|\Psi\rangle = \langle +;\phi|\Psi\rangle |+;\phi\rangle + \langle -;\phi|\Psi\rangle |-;\phi\rangle, \tag{13}$$

where the explicit phase dependence of states $|\pm\rangle$ has been included. The coefficients of expansion can be found easily

$$|\langle +;\phi|\Psi\rangle|^2 = \frac{1}{2}(1 + \cos(u-\phi)\sin(2\theta))$$

$$|\langle -;\phi|\Psi\rangle|^2 = \frac{1}{2}(1-\cos(u-\phi)\sin(2\theta)). \quad (14)$$

Then for any initial atomic state $|\Psi\rangle$ Eq. (11) can be employed for finding the total state of the system at later time. We are concerned with the steady state micromaser field and its phase in particular, therefore we shall need the evolution of the field density matrix $\hat{\rho}$. The initial density matrix of the system $\hat{\rho}_{\text{system}} = |v\rangle|\Psi\rangle\langle\Psi|\langle v|$ evolves in time according Eq. (11) too, of course, and it provides the field density operator after we take trace with respect to the atomic degrees of freedom. Then we obtain

$$|v\rangle\langle v| \to |\langle +;\phi|\Psi\rangle|^2 |\mathrm{e}^{-\mathrm{i}\delta\phi}v\rangle\langle \mathrm{e}^{-\mathrm{i}\delta\phi}v| + |\langle -;\phi|\Psi\rangle|^2 |\mathrm{e}^{\mathrm{i}\delta\phi}v\rangle\langle \mathrm{e}^{\mathrm{i}\delta\phi}v|, \quad (15)$$

where $\delta\phi = g\tau/2\sqrt{\bar{n}}$. In order to find the steady state phase distribution of the field, the P-representation of the density matrix may be used with advantage. The P-representation expresses $\hat{\rho}$ in the diagonal terms in the basis of the coherent states $|\xi\rangle$ as $\hat{\rho} = \int d^2\xi P(\xi)|\xi\rangle\langle\xi|$, which tells us how to convert Eq. (15) into an equation for $P(v)$

$$P(v) \to |\langle\Psi|+;\phi-\delta\phi\rangle|^2 P(v\mathrm{e}^{\mathrm{i}\delta\phi}) + |\langle\Psi|-;\phi+\delta\phi\rangle|^2 P(v\mathrm{e}^{-\mathrm{i}\delta\phi}). \quad (16)$$

It is apparent that the Eq. (16) describes merely changes of the field phase, it does not say how the field amplitude varies with respect to time, so that we drop the explicit $|v|$ dependence in $P(|v|\mathrm{e}^{-\mathrm{i}\phi})$ and then with the help of Eq. (14) we get

$$\begin{aligned} P(\phi) \quad &\to \quad \frac{1}{2}(1+\cos(u-\phi+\delta\phi)\sin 2\theta)P(\phi-\delta\phi) \\ &+ \quad \frac{1}{2}(1-\cos(u-\phi-\delta\phi)\sin 2\theta)P(\phi+\delta\phi). \end{aligned} \quad (17)$$

The right hand side of Eq. (17) must be equal to $P(\phi)$ should we have the steady state field. For the large $\bar{n}$ we have $\delta\phi << 1$, therefore we expand the rhs of Eq. (17) up to the second order of $\delta\phi$ and then the steady state condition give us the second order differential equation for $P(\phi)$

$$0 = -\frac{d}{d\phi}[P(\phi)\cos(u-\phi)\sin 2\theta] + \frac{1}{2}\frac{d^2}{d\phi^2}P(\phi)\delta\phi, \quad (18)$$

which can be integrated providing periodic solutions

$$P(\phi) = C'\mathrm{e}^{-\frac{2\sin 2\theta}{\delta\phi}\sin(u-\phi)}, \quad (19)$$

where C' is the constant of integration. The solutions given by Eq. (19) allow us to make couple of conclusions. Firstly, without even knowing C', one can immediately find the position of the maximum of the phase distribution

$$\phi_{\text{max}} = u - \frac{3\pi}{2}. \quad (20)$$

Secondly, the phase probability distribution $R(\lambda) \equiv \langle\lambda|\hat{\rho}|\lambda\rangle$ can be calculated from

$$R(\lambda) = \int d^2\xi P(\xi)|\langle\xi|\lambda\rangle|^2. \quad (21)$$

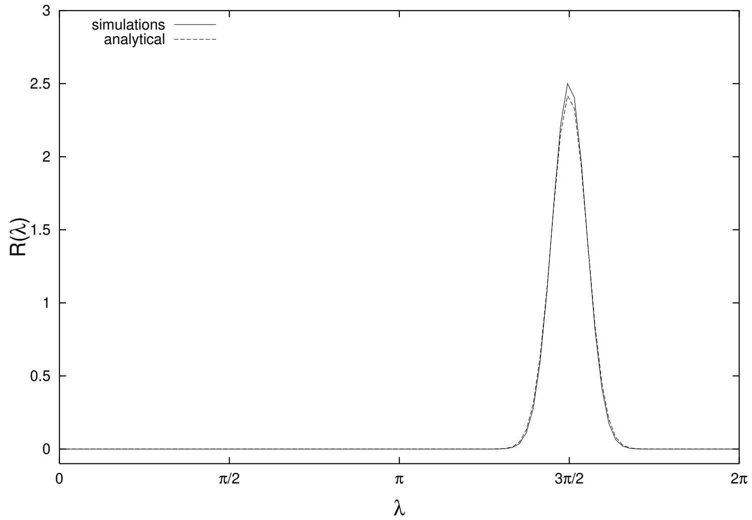

Fig. 1. Phase distribution of the cavity field. Data from the numerical simulations is plotted by the continuous line. Dashed line shows results from Eq. (23).

The coherent state in the number-state representation is defined as

$$|\xi\rangle \equiv \mathrm{e}^{-\frac{|\xi|^2}{2}} \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!}} |n\rangle, \tag{22}$$

and then from Eq. (21) we obtain final expression for $R(\lambda)$ ($r = \sqrt{\bar{n}}$)

$$R(\lambda) = \frac{C}{2\pi} \mathrm{e}^{-r^2} \sum_{n,m=0}^{\infty} \frac{r^{m+n}}{\sqrt{m!n!}} \int_0^{2\pi} d\phi \cos((\phi + \lambda)(n - m)) \mathrm{e}^{-\frac{2\sin 2\theta}{\delta\phi}\sin(u-\phi)}. \tag{23}$$

The constant C is determined from the normalization condition

$$\int_0^{2\pi} R(\lambda) d\lambda = 1. \tag{24}$$

In order to check the validity of Eq. (23) we first ran the numerical simulations. As the initial state of the field was taken the vacuum, the pumping atoms given by Eq. (12) had $u = 0$ and $\sin\theta = 0.9$. The other values of the micromaser parameters were chosen to be as close as possible to those that occur in actual experiments [8]. The atom-field coupling constant g was set to 4.4×10^4 Hz, the time between two consecutive atoms t was set to 6.66×10^{-3} s, the cavity loss coefficient γ was set to $5\ \mathrm{s}^{-1}$, which provided number of atoms passing through the cavity during a single decay time $N_{\mathrm{ex}} = 30$. Then we made a series of trials in order to determine which value of the atom-field interaction time τ gives the largest $\bar{n}$. During the trials we first let the cavity field interact with the pumping atom according to Eq. (5), then followed by a period when the field decayed into the environment, Eq. (7). The measurement of the field was performed just before next atom entered the cavity. When the field reached its steady-state, the mean photon number $\bar{n}$ was measured. It was found that we get maximum $\bar{n} = 23.6$ when $\tau = 5.58 \times 10^{-6}$ s. The values of $\bar{n}$, τ were used for calculating $\delta\phi = 0.02525$. Then we employed Eq. (23) where we set the upper limit for indices m, n to 50. On Fig. 1 we provide both results, from the numerical simulations and analytical, they coincide very well. Also, the location of the maximum of the distribution corresponds to Eq. (20).

Experimenting more with the parameters we found that Eq. (23) provides excellent results having large $\bar{n}$. For the lower values of $\bar{n}$ the peak became less pronounced and the values started to depart from those obtained by simulations.

4 Discussion

Our result for the phase distribution of the steady state micromaser field is in very good correspondence with the numerical simulations in the region of validity of the semiclassical approximation, i.e. when mean photon number $\bar{n}$ is large. The result, however, does not provide information about the amplitude of the field, which remains to be motivation for further research.

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