acta physica slovaca vol. 49 No. 5, 889 – 900 **October 1999**

MATCHED FAST HAAR WAVELET TRANSFORM AND ITS APPLICATION FOR COINCIDENCE NUCLEAR SPECTRA COMPRESSION

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Received 4 May 1999, accepted 1 July 1999

The paper describes a new fast matched Haar wavelet transform aimed for the use in the nuclear spectra compression. The proposed transform is compared with other classical transforms. The results obtained prove in favour of the matched Haar wavelet transform.

PACS: 29.85.+c, 02.60.-x, 07.05.Kf, 02.30.Qy

1 Introduction

Nuclear data measurements are oriented toward gathering a large amount of data. The memory requirements to store multiparameter nuclear spectra can become enormous. For example, if we assume that the content of each channel is represented by one byte, the storage of a four-dimensional histogram with 4096 channels per axis would require a total of 256 terabytes of memory, while the capacity of the largest commercially available hard discs is a few gigabytes. To get around this situation one needs to apply a compression method. The basic premise of successful application of an orthogonal transform in the data compression is a suitable energy distribution in the transform domain. The goal is to concentrate the signal energy as much as possible around the null point of the coordinate system in the transform domain. Hence the key to the good data compression is the orthogonal transform which concentrates signals of a given class in such an efficient way.

Karhunen-Loève transform (KLT) is optimal among all orthogonal transforms in the sense of minimum mean square error [1]. It completely decorrelates the sequence in the transform domain. This enables to process independently one transform coefficient without affecting the others. However the non-existence of fast algorithm and enormous computational complexity for large signal sizes make its use practically impossible.

In terms of variance distribution the Discrete Cosine Transform (DCT) appears to be the near optimal transform for data compression and bandwidth reduction. The DCT is asymptotically equivalent to the KLT [2].

Because the Walsh-Hadamard Transform (WHT) is binary valued its implementation is very simple. Algorithms of WHT require only addition (subtraction) as compared to the complex

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arithmetic operations (complex multiplications and additions) required by DCT. The WHT is frequently used for the data compression.

The fastest and simplest among all orthogonal transforms is Haar based wavelet transform (HWT). Moreover the convergence features of the expansion in Haar functions are superior to the Walsh functions [3], [4].

Traditional transform methods decompose a signal onto a basis derived from some affiliate functions. They do not reflect in any way the shape of compressed signal. WHT and HWT are well suited to compress signals with dominant rectangular components whereas the DCT is well suited to compress signals with dominant sine or cosine components. Whereas the DCT employs sinusoidal waveforms the HWT takes bases derived from a mother wavelet. Wavelet transforms [5] have been used as an effective technique for high efficiency image coding and for studying the spectral behavior of signals [6]. We start with fast Haar transform algorithm and change the coefficients of the signal flow graph. The transform is matched to the mother wavelet.

The arguments presented so far have initiated the idea to retain advantageous Haar structure of transform and to modify its base functions as to achieve maximum possible compression of multidimensional signals with minimum distortion. The coefficients of Haar transform are modified based on the criterion of maximum compression with respect to a given data class, which is represented for each dimension by one mother wavelet vector. Hereafter these vectors are called reference vectors.

2 Construction of matched Haar wavelet transform

Let the vector $\mathbf{x}$ be a mother wavelet vector. Without loss of generality we shall illustrate the algorithm for $N = 4$. Then the conditions to set up the transform matrix will be generalized for any size which is power of 2. The Fig. 1 shows the signal flow chart for the fast matched Haar wavelet transform (MHWT). We have denoted the multiplication coefficients as $a_i, b_i, c_i, d_i, i \in \{0, 1, 2\}$.

Then the output vector is

$$\mathbf{y} = \mathbf{T} \cdot \mathbf{x}, \tag{1}$$

where $\mathbf{T}$ is transform matrix given, according to Fig. 1, as

$$\mathbf{T} = \begin{bmatrix} a_0a_2 & a_1c_2 & c_0a_2 & c_1c_2 \\ a_0b_2 & -a_1d_2 & c_0b_2 & -c_1d_2 \\ \frac{b_0}{\sqrt{2}} & 0 & \frac{-d_0}{\sqrt{2}} & 0 \\ 0 & \frac{b_1}{\sqrt{2}} & 0 & \frac{-d_1}{\sqrt{2}} \end{bmatrix}. \tag{2}$$

The transform matrix should be orthonormal, so that

$$\mathbf{T} \cdot \mathbf{T}^{\mathbf{T}} = \mathbf{E}, \tag{3}$$

where $\mathbf{E}$ denotes the unit matrix. Then the conditions of orthonormality are

$$\begin{aligned} \left(a_0^2 + c_0^2\right)a_2^2 + \left(a_1^2 + c_1^2\right)c_2^2 &= 1, \\ \left(a_0^2 + c_0^2\right)a_2b_2 - \left(a_1^2 + c_1^2\right)c_2d_2 &= 0, \end{aligned}$$

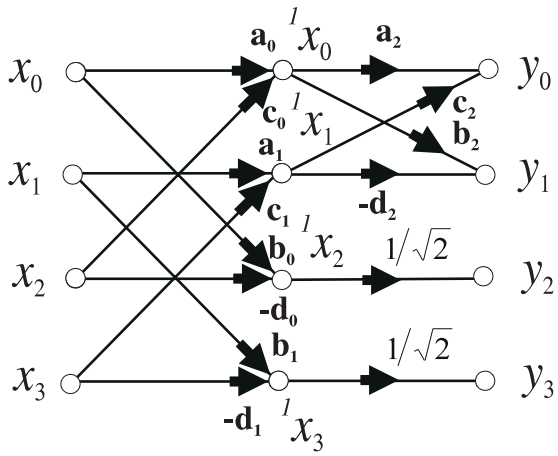

Fig. 1. Signal flow graph of the matched fast Haar wavelet transform (N=4)

$$
\begin{aligned}
(a_0b_0 - c_0d_0)\frac{a_2}{\sqrt{2}} &= 0,\\
(a_1b_1 - c_1d_1)\frac{c_2}{\sqrt{2}} &= 0,\\
(a_0^2 + c_0^2)a_2b_2 - (a_1^2 + c_1^2)c_2d_2 &= 0,\\
(a_0^2 + c_0^2)b_2^2 + (a_1^2 + c_1^2)d_2^2 &= 1,\\
(a_0b_0 - c_0d_0)\frac{b_2}{\sqrt{2}} &= 0,\\
-(a_1b_1 - c_1d_1)\frac{d_2}{\sqrt{2}} &= 0,\\
(a_0b_0 - c_0d_0)\frac{a_2}{\sqrt{2}} &= 0,\\
(a_0b_0 - c_0d_0)\frac{b_2}{\sqrt{2}} &= 0,\\
\frac{b_0^2 + d_0^2}{2} + \frac{b_1^2 + d_1^2}{2} &= 1,\\
(a_0b_0 - c_0d_0)\frac{c_2}{\sqrt{2}} &= 0,\\
(a_1b_1 - c_1d_1)\frac{d_2}{\sqrt{2}} &= 0,\\
\frac{b_0^2 + d_0^2}{2} + \frac{b_1^2 + d_1^2}{2} &= 1.
\end{aligned}
\tag{4}
$$

The conditions (4) are satisfied if and only if

$$a_i b_i - c_i d_i = 0 \tag{5a}$$

$$a_i^2 + c_i^2 = 1 \tag{5b}$$

$$b_i^2 + d_i^2 = 1, \tag{5c}$$

where $i = 0, \ldots, 3$. From (5a) substituting

$$a_i = \frac{c_i d_i}{b_i} \tag{5d}$$

into (5c) one gets

$$b_i^2 = c_i^2$$

and thus

$$b_i = \pm c_i. \tag{5e}$$

By back substitution from (5e) into (5d) we have

$$a_i = \pm d_i. \tag{5f}$$

These are necessary and sufficient conditions ensuring the orthonormality of transform matrix of the fast MHWT. For each element of the fast transform (butterfly) it must hold that the input energy is equal to the output energy (Parseval's theorem). Hence for the first butterfly we obtain

$$x_0^2 + x_2^2 = {}^1x_0^2 + {}^1x_2^2. \tag{6}$$

One of the ways to predetermine the distribution of output energy is its concentration into one output element. Our goal is to achieve maximum possible compression. Then we have

$$\begin{aligned} {}^1x_0 &= \sqrt{x_0^2 + x_2^2}, \\ {}^1x_2 &= 0. \end{aligned} \tag{7}$$

In general, taking (5e), (5f) with the positive sign, one can write

$$\begin{aligned} a_0 x_0 + b_0 x_2 &= {}^1x_0, \\ b_0 x_0 - a_0 x_2 &= {}^1x_2. \end{aligned} \tag{8}$$

Substituting (8) into (7) yields

$$a_0^2 + b_0^2 = 1. \tag{9}$$

Using (5), (6), (7), (9) we finally obtain

$$\begin{aligned} a_0 &= \frac{x_0}{\sqrt{x_0^2 + x_2^2}} & b_0 &= \frac{x_2}{\sqrt{x_0^2 + x_2^2}}, \\ d_0 &= a_0 & c_0 &= b_0. \end{aligned} \tag{10}$$

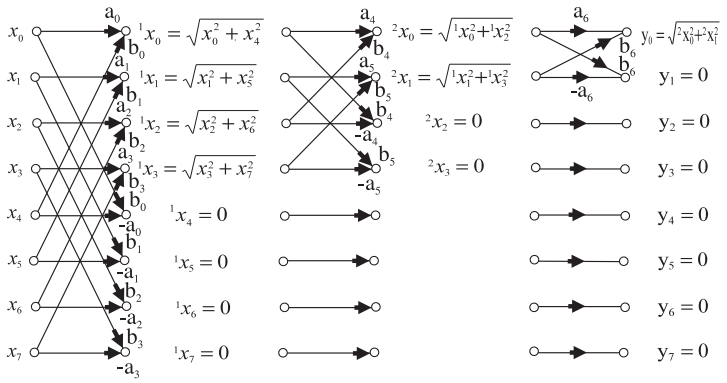

Fig. 2. Signal flow graph of the matched fast Haar wavelet transform (N=8)

From the point of view of maximum compression these are the optimal transform coefficients of one butterfly of the matched Haar wavelet transform.

Let us proceed larger length. Let us take N=8 and calculate the first iteration step of the fast transform (Fig. 2) The butterfly coefficients in the first iteration step are given by (10)

$$a_i = \frac{\sqrt{x_i^2}}{\sqrt{x_i^2 + x_{i+4}^2}}, \qquad b_i = \frac{\sqrt{x_{i+4}^2}}{\sqrt{x_i^2 + x_{i+4}^2}}, \tag{11}$$

$$c_i = b_i \qquad d_i = -a_i,$$

where $i = 0, 1, 2, 3$. With regard to the Fig. 2 and (10) in the next iteration step one obtains

$$a_{i+4} = \frac{\sqrt{x_i^2 + x_{i+4}^2}}{\sqrt{x_i^2 + x_{i+2}^2 + x_{i+4}^2 + x_{i+6}^2}}, \qquad b_{i+4} = \frac{\sqrt{x_{i+2}^2 + x_{i+6}^2}}{\sqrt{x_i^2 + x_{i+2}^2 + x_{i+4}^2 + x_{i+6}^2}}, \tag{12}$$

$$c_{i+4} = b_{i+4} \qquad d_{i+4} = a_{i+4},$$

where $i = 0, 1$. Analogously, in the third iteration step we set the coefficients a, b, c, d

$$a_6 = \frac{\sqrt{x_0^2 + x_2^2 + x_4^2 + x_6^2}}{\sqrt{\sum_{i=0}^{7} x_i^2}}, \qquad b_6 = \frac{\sqrt{x_1^2 + x_3^2 + x_5^2 + x_7^2}}{\sqrt{\sum_{i=0}^{7} x_i^2}}, \tag{13}$$

$$c_6 = b_6, \qquad d_6 = a_6.$$

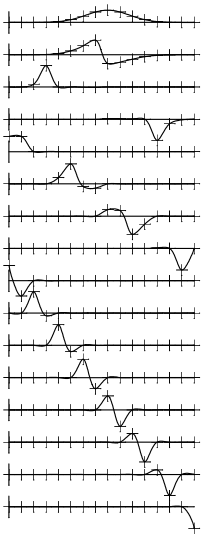

Fig. 3. Example of base functions of the matched Haar transform ($N = 16$)

The algorithm to calculate the butterfly coefficients for larger transform lengths is analogous from the previous procedure. Generally one can write the transform coefficients

$$a_{l+i} = \frac{\sqrt{\sum_{j=0}^{2^{s-1}-1} x^2_{j\cdot\frac{N}{2^{s-1}}+i}}}{\sqrt{\sum_{j=0}^{2^s-1} x^2_{j\cdot\frac{N}{2^s}+i}}} \qquad b_{l+i} = \frac{\sqrt{\sum_{j=0}^{2^{s-1}-1} x^2_{j\cdot\frac{N}{2^{s-1}}+\frac{N}{2^s}+i}}}{\sqrt{\sum_{j=0}^{2^s-1} x^2_{j\cdot\frac{N}{2^s}+i}}},$$
$$c_{l+i} = b_{l+i}, \qquad d_{l+i} = a_{l+i}, \tag{14}$$

where s denotes iteration step, $s \in \{1, \ldots, \log_2 N\}$, i denotes position of coefficients in the given iteration step, $i \in \{0, \ldots, N/2^s - 1\}$,

$$l = \sum_{k=1}^{s-1} \frac{N}{2^k}, \tag{15}$$

and N is the length of the transform. Using multiplication coefficients (14), (15) one can calculate inverse matched Haar wavelet transform by applying reverse ordered signal flow graph.

For the sake of illustration we have generated the base functions of MHWT for $N = 16$ and the mother wavelet vector (gaussian) defined as $\exp\{\frac{-(i-8)^2}{6}\}$ (Fig. 3).

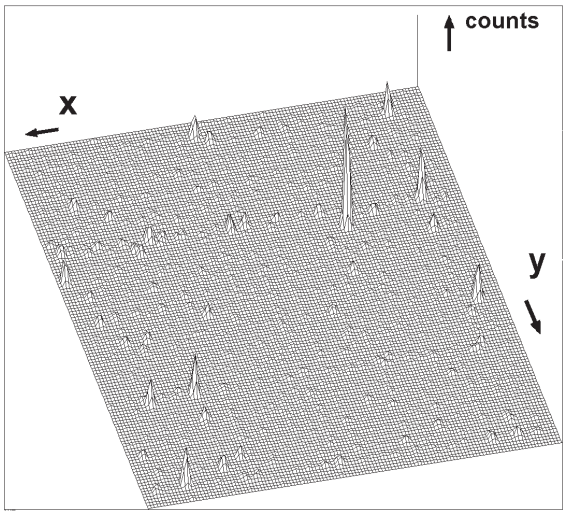

Fig. 4. Original 2-dimensional signal (coincident nuclear spectrum)

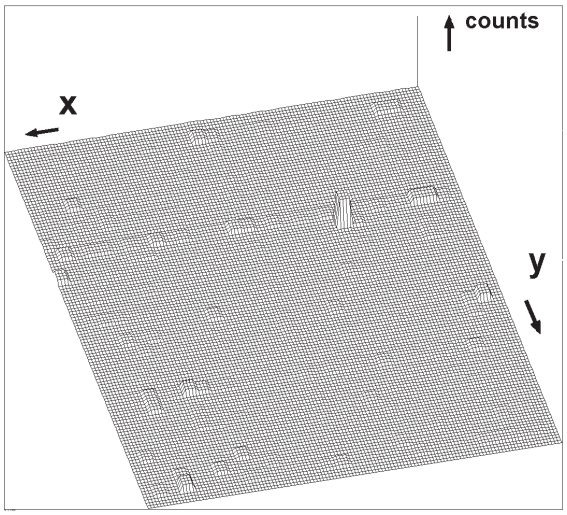

Fig. 5. 2-dimensional nuclear spectrum after the compression using classical Haar transform with $CR = 64$

3 Experimental results

Before using the MHWT must be identified, i.e., the multiplication coefficients of the fast algorithm of MHWT must be determined. The transform coefficients for each dimension are cal-

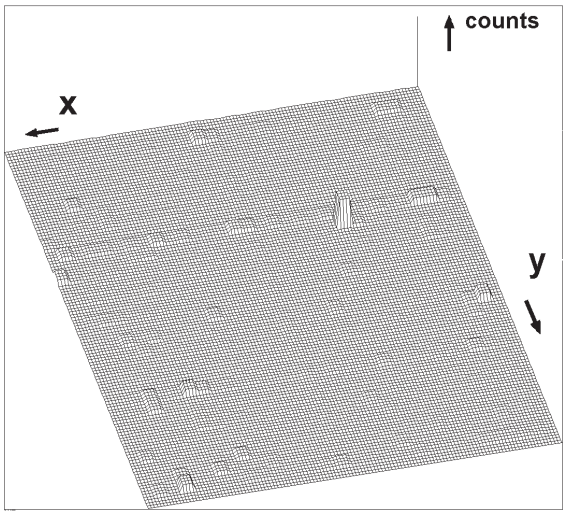

Fig. 6. 2-dimensional nuclear spectrum after the compression using Walsh-Hadamard transform with $CR = 64$

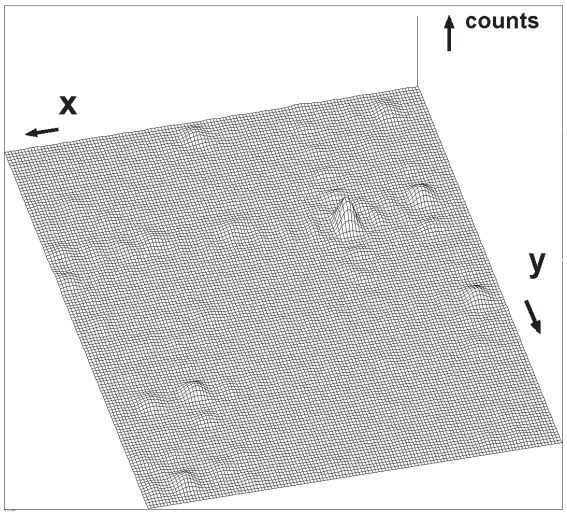

Fig. 7. 2-dimensional nuclear spectrum after the compression using Cosine transform with $CR = 64$

culated from the mother wavelet vector, which represents in a way the compressed signal. In the multidimensional transform we shall employ for each dimension one mother wavelet vector. For each dimension in an M-dimensional signal we calculate integral vector, which is used to be

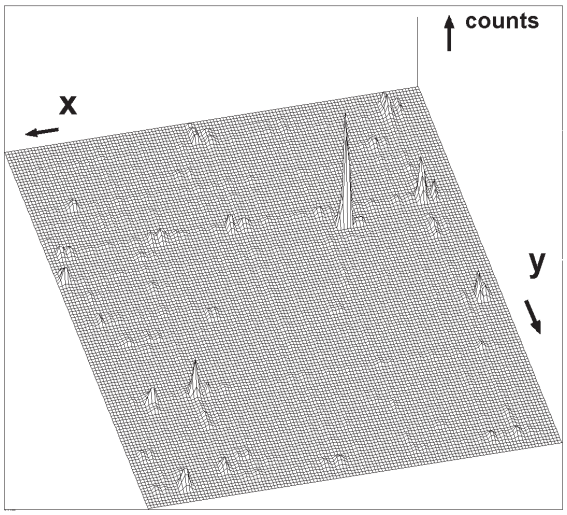

Fig. 8. 2-dimensional nuclear spectrum after the compression using matched Haar wavelet transform with $CR = 64$

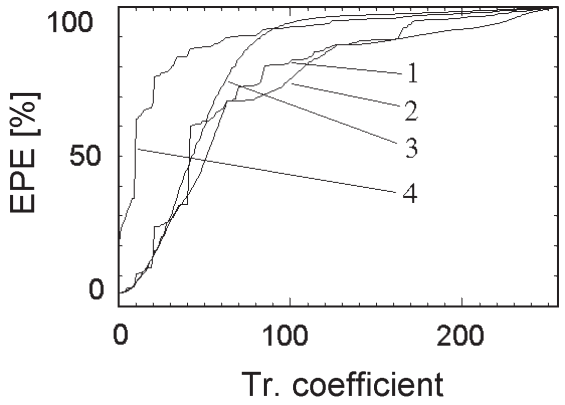

Fig. 9. EPE functions for the spectrum from the Fig. 4 and the following transforms: 1-classical Haar transform, 2-Walsh-Hadamard transform, 3-Cosine transform, 4-matched Haar wavelet transform

mother wavelet vector for identification of MHWT

$$\begin{aligned} S_1(i_1) &= \sum_{i_2=0}^{N_2-1} \sum_{i_3=0}^{N_3-1} \cdots \sum_{i_M=0}^{N_M-1} x(i_1, i_2, \cdots, i_M) \\ S_2(i_2) &= \sum_{i_1=0}^{N_1-1} \sum_{i_3=0}^{N_3-1} \cdots \sum_{i_M=0}^{N_M-1} x(i_1, i_2, \cdots, i_M) \\ &\vdots \\ S_M(i_M) &= \sum_{i_1=0}^{N_1-1} \sum_{i_2=0}^{N_2-1} \cdots \sum_{i_{M-1}=0}^{N_{M-1}-1} x(i_1, i_2, \cdots, i_M), \end{aligned} \tag{16}$$

where $i_1, i_2, \ldots, i_M \in \{0, 1, \ldots, N-1\}$.

This way of construction of mother wavelet vectors proved to be the most efficient. In what follows we present the comparison of compression efficiency and quality of different transforms. For the illustration we have chosen a part of nuclear spectra matrix of nuclear spectra of $\gamma - \gamma$ coincidences [7]. In the first example (Fig. 4) we have chosen a nuclear spectrum of the size 256x256 points. This spectrum of fission fragment pairs has been collected with the Gammasphere spectrometer in experiment in LBNL Berkeley [8]. The spectrum contains the transitions belonging to the partner fission fragments. We shall employ classical Haar, Walsh, Cosine and matched Haar wavelet transforms with the same compression ratio $CR = 64$. Figs. 5, 6 show the nuclear spectrum from the Fig. 4 after the compression via classical Haar transform and Walsh transform, respectively. One can observe the great distortion of the signal and dominant rectangular components. Fig. 7 presents the same signal however now after the compression via Cosine transform. The rectangular structure of the signal disappeared. However comparing with the Fig. 4 obviously some details are lost though the signal is smooth. The details remained preserved when using MHWT (Fig. 8). In the example presented apparently the quality of the compression via MHWT is the best.

In [9] the energy packing efficiency is introduced. It is defined as part of energy contained in the first M from N transform coefficients

$$EPE(M) = \frac{\sum_{i=0}^{M-1} y_i^2}{\sum_{i=0}^{N-1} y_i^2} \cdot 100[\%]. \tag{17}$$

Since the energy in the transform domain of two-dimensional (also multidimensional) signals is concentrated around the null point of the coordinate system we shall define analogously the energy packing efficiency for two-dimensional transforms

$$EPE(M) = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} y_{i,j}^2}{\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} y_{i,j}^2} \cdot 100[\%]. \tag{18}$$

The Fig. 9 shows the functions $EPE(M)$ for the signal from the Fig. 4 and above mentioned transforms in dependence on transform coefficient $\mathbf{y}$ defined in (1). One can observe that the matched Haar wavelet transform concentrates the signal energy in the region of low transform coefficients. From the point of view of compression it is the most efficient transform from all above presented transforms.

4 Conclusions

In the paper the algorithm of new fast matched Haar wavelet transform aimed for signal compression has been defined. The method is based on a direct modification of multiplicative coefficients

of the signal flow graph of the fast Haar transform algorithm. The paper shows that principle of adaptability can be also applied to the fast Haar wavelet transform signal flow graph. The transform matrix is constructed according to the mother wavelet vector representing the processed nuclear spectra. Setting the vectors equal to integral signals has shown to be the most suitable for construction of the transform matrix for compression of nuclear spectra. However for other kind of nuclear spectra the other methods of choice of the mother wavelet vectors are also possible.

The presented method has been used to compress nuclear spectra measured in nuclear physics experiments. In the examples presented we have compared it to the classical transforms. The figures as well as EPE functions show that the proposed way of compression gives evidently better results.

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