

**ENTANGLEMENT TRANSFORMATION AT ABSORBING AND AMPLIFYING DIELECTRIC FOUR-PORT DEVICES\*****S. Scheel<sup>1</sup>, L. Knöll, T. Opatrný, D.-G. Welsch***Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena,  
Max-Wien-Platz 1, D-07743 Jena, Germany*

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Dielectric four-port devices play an important role in optical quantum information processing. Since for causality reasons the permittivity is a complex function of frequency, dielectrics are typical examples of noisy quantum channels, which prevent them from preserving quantum coherence. To study the effects of quantum decoherence, we start from the quantized electromagnetic field in an arbitrary Kramers–Kronig dielectric of given complex permittivity and construct the transformation that relates at a four-port device the output quantum state to the input quantum state, without placing restrictions on the frequency. Basing on the relative entropy as an entanglement measure, we apply the formalism to the transformation of entanglement, with special emphasis on the entanglement degradation in absorbing optical fibers. In particular we show that the Bell basis states  $|\Psi^\pm\rangle$  using Fock states are more robust against decoherence than the states  $|\Phi^\pm\rangle$ .

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**1 Introduction**

Quantum communication schemes widely use dielectric four-port devices as basic elements for constructing optical quantum channels (Fig. 1). Since for causality reasons the permittivity is necessarily a complex function of frequency, dielectrics are typical examples of noisy quantum channels, in which quantum coherence is not preserved. To study the effects of decoherence, we present an approach based on the quantization of the electromagnetic field in linear Kramers-Kronig dielectrics. It replaces the familiar mode decomposition with a source-quantity representation, where the field is expressed in terms of the classical dyadic Green function and fundamental variables of the composed field-matter system. An advantage of the method is that it is universally valid as long as the medium can be characterized in terms of a spatially varying complex permittivity. In this way, it enables us to construct the transformation relating the output quantum state to the input quantum state at absorbing four-port devices in terms of the actual device parameters, without placing frequency restrictions and without using replacement schemes.

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<sup>1</sup>E-mail address: scheel@tpi.uni-jena.de

Knowing the full transformed quantum state, we can then compute the entanglement contained in the output quantum state. In particular, applying the formalism to optical fibers, we show that the Bell-type basis states  $|\Psi\rangle$  are more robust against decoherence than the states  $|\Phi\rangle$ .

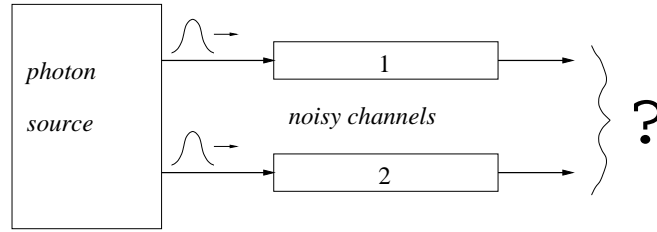


Fig.1. Schematic quantum communication experiment. The photon source produces two-mode entangled, each mode being transmitted through a noisy channel, e.g., an optical fiber.

Our procedure is divided into two subsequent steps. First we have to know how to quantize the electromagnetic field in causal dielectrics. It will give us a representation of the field operators in terms of the material properties and its geometry. Second, an open-systems approach is employed to compute the transformed quantum state exactly. The basic idea is here to enlarge the Hilbert space of the field by the Hilbert space of the device (including the dissipative system), then performing a unitary transformation in it and projecting back onto the subspace of the field.

## 2 Quantum-state transformation

Let us consider the first step in our procedure. We restrict ourselves to a quasi one-dimensional scheme, as depicted in Fig. 2, in which the dielectric device is surrounded by vacuum. Applying the formalism developed in [1], we quantize the electromagnetic field in the presence of the device by means of a Green function representation of the field and introduction of bosonic vector fields playing the role of the collective excitations of the field, the dielectric matter, and the reservoir [1]. It turns out that outside the device the usual mode expansion applies, the  $\hat{a}_i(\omega)$  and  $\hat{b}_i(\omega)$  in Fig. 2 being respectively the photonic operators of the incoming and outgoing plane waves at frequency  $\omega$ .

It then follows that the action of the dielectric device on the incoming radiation can be described by quantum optical input-output relations [2] which, in fact, are nothing but a suitable rewriting of the corresponding one-dimensional Green function. Let  $\hat{g}_i(\omega)$  be the bosonic operators of the device excitations which play the role of noise forces associated with absorption, and introduce the two-vector notation  $\hat{\mathbf{a}}(\omega)$ ,  $\hat{\mathbf{b}}(\omega)$ , and  $\hat{\mathbf{g}}(\omega)$  for the field and device operators respectively. The input-output relations can then be written in a compact form using the characteristic transmission and absorption matrices  $\mathbf{T}(\omega)$  and  $\mathbf{A}(\omega)$ , respectively, derived in [2]:

$$\hat{\mathbf{b}}(\omega) = \mathbf{T}(\omega)\hat{\mathbf{a}}(\omega) + \mathbf{A}(\omega)\hat{\mathbf{g}}(\omega), \quad (1)$$

where the energy-conservation relation

$$\mathbf{T}(\omega)\mathbf{T}^+(\omega) + \mathbf{A}(\omega)\mathbf{A}^+(\omega) = \mathbf{I} \quad (2)$$

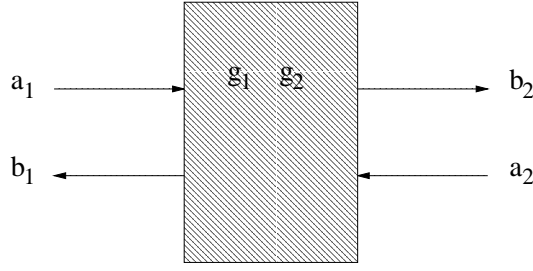


Fig.2. Quasi one-dimensional geometry of the dielectric device with input modes  $\hat{a}_i(\omega)$ , output modes  $\hat{b}_i(\omega)$  and device excitations  $\hat{g}_i(\omega)$ .

is satisfied. Eqs. (1) and (2) are valid for any frequency. Hence, knowledge of the input operators as functions of frequency allows us to construct the output operators.

The second step in the quantum-state transformation now consists of the open-systems approach. Suppose the incoming field is prepared in some state of the Hilbert space  $\mathcal{H}_{\text{field}}$  and the device (including the reservoir) is initially prepared in some state of the Hilbert space  $\mathcal{H}_{\text{device}}$ , without correlations between them. Then, before the quantum-state transformation the full Hilbert space is just the tensor product  $\mathcal{H}_{\text{field}} \otimes \mathcal{H}_{\text{device}}$ . As we will see, in that enlarged space a unitary operator can be constructed, whereas in the space  $\mathcal{H}_{\text{field}}$  it could not due to the dissipation processes.

Let us define the four-vector operators

$$\hat{\alpha}(\omega) = \begin{pmatrix} \hat{\mathbf{a}}(\omega) \\ \hat{\mathbf{g}}(\omega) \end{pmatrix}, \quad \hat{\beta}(\omega) = \begin{pmatrix} \hat{\mathbf{b}}(\omega) \\ \hat{\mathbf{h}}(\omega) \end{pmatrix} \quad (3)$$

with some auxiliary bosonic (two-vector) device operator  $\hat{\mathbf{h}}(\omega)$ . Then, the input-output relation (1) can be extended to a four-dimensional transformation

$$\hat{\beta}(\omega) = \mathbf{\Lambda}(\omega) \hat{\alpha}(\omega), \quad (4)$$

with  $\mathbf{\Lambda}(\omega) \in \text{SU}(4)$  [3]. Explicitly,

$$\mathbf{\Lambda}(\omega) = \begin{pmatrix} \mathbf{T}(\omega) & \mathbf{A}(\omega) \\ -\mathbf{S}(\omega)\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{C}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega) \end{pmatrix} \quad (5)$$

with the positive commuting Hermitian matrices

$$\mathbf{C}(\omega) = \sqrt{\mathbf{T}(\omega)\mathbf{T}^+(\omega)}, \quad \mathbf{S}(\omega) = \sqrt{\mathbf{A}(\omega)\mathbf{A}^+(\omega)}. \quad (6)$$

Hence, there is a unitary operator transformation

$$\hat{\beta}(\omega) = \hat{U}^\dagger \hat{\alpha}(\omega) \hat{U} \quad (7)$$

where

$$\hat{U} = \exp \left\{ -i \int d\omega \left[ \hat{\alpha}^\dagger(\omega) \right]^T \mathbf{\Phi}(\omega) \hat{\alpha}(\omega) \right\} \quad (8)$$

and

$$\mathbf{\Lambda}(\omega) = e^{-i\Phi(\omega)}. \quad (9)$$

Note that  $\hat{U}$  acts in the product space  $\mathcal{H}_{\text{field}} \otimes \mathcal{H}_{\text{device}}$ . Given a density operator  $\hat{\rho}_{\text{in}}$  of the input quantum state as a functional of  $\hat{\alpha}(\omega)$ , the density operator of the output quantum state is obtained by a unitary transformation with the operator  $\hat{U}$  from Eq. (8) and projecting back onto the Hilbert space  $\mathcal{H}_{\text{field}}$ . Hence,

$$\hat{\rho}_{\text{out}}^{(\text{F})} = \text{Tr}^{(\text{D})} \left\{ \hat{U} \hat{\rho}_{\text{in}} \hat{U}^\dagger \right\} = \text{Tr}^{(\text{D})} \left\{ \hat{\rho}_{\text{in}} \left[ \mathbf{\Lambda}^+(\omega) \hat{\alpha}(\omega), \mathbf{\Lambda}^T(\omega) \hat{\alpha}^\dagger(\omega) \right] \right\}, \quad (10)$$

where  $\text{Tr}^{(\text{D})}$  means tracing with respect to the device variables. Note, that the difference to usually considered open-systems theories is provided by the fact that we actually know how the dissipative environment (e.g., a dispersing and absorbing fiber) acts on our quantum states.

Finally, let us make some remarks about amplifying media. In contrast to Eq. (1), we have to insert the noise *creation* operators  $\hat{g}_i^\dagger(\omega)$  into the input-output relation. The energy conservation relation (2) then changes to

$$\mathbf{T}(\omega) \mathbf{T}^+(\omega) - \mathbf{A}(\omega) \mathbf{A}^+(\omega) = \mathbf{I}. \quad (11)$$

The corresponding  $4 \times 4$ -matrix  $\mathbf{\Lambda}(\omega)$  will then be a group element of the non-compact group  $\text{SU}(2,2)$ .

### 3 Entanglement transformation

We now turn to the problem of entanglement transformation, restricting our attention to two discrete (quasimonochromatic) modes. In order to quantify the entanglement of the modes, we use a measure  $E(\hat{\sigma})$  based on the quantum relative entropy and define the amount of entanglement by the distance of the density matrix under examination to the set of all separable density matrices [4],

$$E(\hat{\sigma}) = \min_{\hat{\rho} \in \mathcal{D}} \text{Tr} [\hat{\sigma} (\ln \hat{\sigma} - \ln \hat{\rho})]. \quad (12)$$

Unfortunately, analytical expressions for the closest separable density matrix are known only in very special cases. Therefore, numerical minimum search in the  $\{N_1^2 N_2^2 [2(N_1 + N_2 - 2) + 1] - 1\}$ -dimensional parameter space is needed, where  $N_1$  and  $N_2$  are the dimensions of the Hilbert spaces of the subsystems. Fortunately, as pointed out in [4], due to convexity of the relative entropy, there can be only one global minimum.

Suppose the two incoming modes in Fig. 1 are prepared in Bell-type state

$$|\Psi_n^\pm\rangle = \frac{1}{\sqrt{2}} (|0n\rangle \pm |n0\rangle). \quad (13)$$

These states are called maximally entangled because they contain exactly  $\ln 2$  entanglement, equivalent to 1 bit. Applying Eq. (10), we obtain [5]

$$\hat{\rho}_{\text{out}}^{(\text{F})} = \lambda \hat{\rho}_{\text{sep}} + (1 - \lambda) |\Psi_n^\pm\rangle \langle \Psi_n^\pm|, \quad (14)$$

where

$$\lambda \hat{\rho}_{\text{sep}} = \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left[ |T_1|^{2k} (1-|T_1|^2)^{n-k} |k0\rangle \langle k0| + |T_2|^{2k} (1-|T_2|^2)^{n-k} |0k\rangle \langle 0k| \right], \quad (15)$$

$$|\Psi'_n{}^\pm\rangle = (|T_1|^{2n} + |T_2|^{2n})^{-1/2} (T_1^n |n0\rangle \pm T_2^n |0n\rangle), \quad (16)$$

and

$$(1-\lambda) = \frac{1}{2} (|T_1|^{2n} + |T_2|^{2n}), \quad (17)$$

( $T_i$ , transmission coefficient of the  $i$ th channel,  $i=1, 2$ ).

We have written the output density matrix (14) as a sum of the separable states (15) and the pure state (16), which gives us the opportunity to estimate the entanglement by employing the convexity property of the relative entropy [6]

$$E[\lambda \hat{\sigma}_1 + (1-\lambda) \hat{\sigma}_2] \leq \lambda E(\hat{\sigma}_1) + (1-\lambda) E(\hat{\sigma}_2). \quad (18)$$

Applying the inequality (18) to the state (14) and observing that separable states do not contribute to the entanglement by definition, we can estimate the entanglement of the state (14) by the entanglement of the pure state (16), which is simply given by the von Neumann entropy of one subsystem. Thus,

$$E(\hat{\rho}_{\text{out}}^{(F)}) \leq \frac{1}{2} \left[ (|T_1|^{2n} + |T_2|^{2n}) \ln (|T_1|^{2n} + |T_2|^{2n}) - |T_1|^{2n} \ln |T_1|^{2n} - |T_2|^{2n} \ln |T_2|^{2n} \right] \quad (19)$$

which for equal transmission  $T_1 = T_2 = T$  reduces to

$$E(\hat{\rho}_{\text{out}}^{(F)}) \leq |T|^{2n} \ln 2. \quad (20)$$

Equation (20) shows that entanglement degradation exponentially increases with the number of photons involved. Such behaviour is typical for quantum interference phenomena [7].

We can apply the same procedure to the Bell-type states

$$|\Phi_n{}^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |nn\rangle). \quad (21)$$

The output density matrix now reads

$$\begin{aligned} \hat{\rho}_{\text{out}}^{(F)} = & \frac{1}{2} \sum_{k_1, k_2=0}^n \binom{n}{k_1} \binom{n}{k_2} |T_1|^{2k_1} |T_2|^{2k_2} (1-|T_1|^2)^{n-k_1} (1-|T_2|^2)^{n-k_2} |k_1 k_2\rangle \langle k_1 k_2| \\ & - \frac{1}{2} |T_1|^{2n} |T_2|^{2n} |nn\rangle \langle nn| + \frac{1}{2} (1 + |T_1 T_2|^{2n}) |\Phi_n{}^\pm\rangle \langle \Phi_n{}^\pm| \end{aligned} \quad (22)$$

with

$$|\Phi_n{}^\pm\rangle = (1 + |T_1 T_2|^{2n})^{-1/2} [ |00\rangle + T_1^n T_2^n |nn\rangle ]. \quad (23)$$

The upper bound for the entanglement is derived to be

$$E(\hat{\rho}_{\text{out}}^{(F)}) \leq \frac{1}{2} \left[ (1 + |T_1 T_2|^{2n}) \ln (1 + |T_1 T_2|^{2n}) - |T_1 T_2|^{2n} \ln |T_1 T_2|^{2n} \right]. \quad (24)$$

For small transmission coefficients, i.e., for large propagation lengths, we can expand Eq. (24) to obtain

$$E(\hat{\rho}_{\text{out}}^{(F)}) \leq \frac{1}{2} |T_1 T_2|^{2n} (1 - \ln |T_1 T_2|^{2n}) . \quad (25)$$

It is instructive to compare the entanglement degradation of the states  $|\Phi_n^\pm\rangle$  with that of the states  $|\Psi_n^\pm\rangle$ . From Eqs. (16) and (23) we see that the probability of finding  $n$  photons in one channel after transmission decreases as  $|T_i|^n$  for the states  $|\Psi_n^\pm\rangle$  but decreases as  $|T_1 T_2|^n$  for the states  $|\Phi_n^\pm\rangle$ . Therefore we expect the entanglement degradation to be faster for the  $|\Phi_n^\pm\rangle$  states than for the  $|\Psi_n^\pm\rangle$  states. Comparing Eqs. (20) and (25) for equal transmission coefficients  $T_1 = T_2 = T \ll 1$ , we find that

$$\frac{E(|\Phi_n^\pm\rangle)}{E(|\Psi_n^\pm\rangle)} \approx \frac{|T|^{2n} (1 - \ln |T|^{4n})}{2 \ln 2} , \quad (26)$$

which indeed confirms our expectation. Figure 3 shows the numerical results for one-photon Bell basis states using a transmission coefficient  $|T| = \exp(-l/l_A)$ , with  $l_A$  being the absorption length.

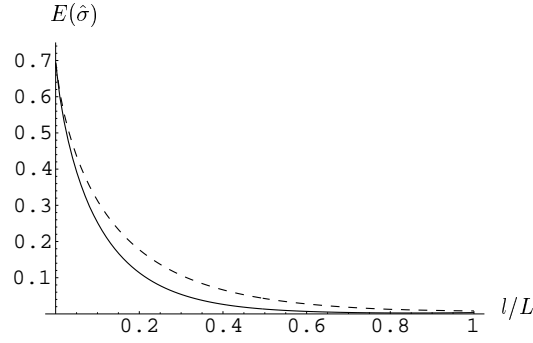


Fig.3. Comparison of entanglement degradation of one-photon Bell basis states  $|\Phi_1^\pm\rangle$  (full curve) and  $|\Psi_1^\pm\rangle$  (dashed curve).

Finally, let us briefly address the two-mode squeezed vacuum

$$|\Psi\rangle = \exp\left[\zeta(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2)\right] |00\rangle = \sqrt{1 - q^2} \sum_{n=0}^{\infty} q^n |nn\rangle \quad (27)$$

( $q = \tanh \zeta$ ,  $\zeta$  real), whose (Gaussian) Wigner function reads

$$W(\boldsymbol{\xi}) = \left(4\pi^2 \sqrt{\det \mathbf{V}}\right)^{-1} \exp\left\{-\frac{1}{2} \boldsymbol{\xi}^T \mathbf{V}^{-1} \boldsymbol{\xi}\right\} , \quad (28)$$

where  $\xi$  is a four-vector whose elements are the quadrature-component variables  $q_1, p_1, q_2, p_2$ , and  $\mathbf{V}$  is the  $4 \times 4$  variance matrix of the Wigner function,

$$\mathbf{V} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} = \begin{pmatrix} c/2 & 0 & -s/2 & 0 \\ 0 & c/2 & 0 & s/2 \\ -s/2 & 0 & c/2 & 0 \\ 0 & s/2 & 0 & c/2 \end{pmatrix} \quad (29)$$

[ $c = \cosh 2\zeta$ ,  $s = \sinh 2\zeta$ ]. Transmitting the two-mode squeezed vacuum through absorbing fibers at some temperature  $\vartheta$ , the Wigner function of the transformed state is again a Gaussian. Using the input-output relations (1), we can easily transform the input variance matrix (29) to obtain the output variance matrix. Application of the Peres-Horodecki separability criterion [8]

$$\begin{aligned} & \det \mathbf{A} \det \mathbf{B} + \left(\frac{1}{4} - |\det \mathbf{C}|\right)^2 - \text{Tr}(\mathbf{A}\mathbf{J}\mathbf{C}\mathbf{J}\mathbf{B}\mathbf{J}\mathbf{C}^T\mathbf{J}) \\ & \geq \frac{1}{4}(\det \mathbf{A} + \det \mathbf{B}), \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \quad (30)$$

to the output variance matrix yields (for equal fibers) the inequality

$$n \geq \frac{|T|^2 (1 - e^{-2|\zeta|})}{2(1 - |R|^2 - |T|^2)} \quad (31)$$

relating the mean number of the fiber excitations,  $n = (\exp \hbar\omega / (k_B\vartheta) - 1)^{-1}$ , to the transmission and reflection coefficients of the fibers. Hence, for chosen squeezing parameter  $\zeta$  and chosen transmission and reflection coefficients  $T$  and  $R$ , respectively, there exists a maximal temperature  $\vartheta$  and correspondingly a maximal mean excitation number of the thermal state the fibers are prepared in, such that the transmitted quantum state is still not separable.

#### 4 Conclusions

We have shown how the formalism of quantization of the electromagnetic field in absorbing dielectrics of given complex permittivity can be used to determine the quantum state of the outgoing modes at an absorbing four-port device and to calculate the amount of entanglement realized at the output ports of the device. Since with increasing dimension of the Hilbert space the (numerical) calculation of the entanglement becomes an effort, we restricted our attention to relatively simple states and tried to derive reasonable analytical estimations of the entanglement.

In particular, we have studied the entanglement degradation in optical fibers of Bell-type basis states  $|\Psi_n^\pm\rangle$  and  $|\Phi_n^\pm\rangle$  containing  $n$  photons. From the derived bounds of the entanglement we have shown that the entanglement decreases exponentially with increasing (initial) photon number at least. Further, we have shown that the entanglement degradation is more rapidly for the states  $|\Phi_n^\pm\rangle$  than for the states  $|\Psi_n^\pm\rangle$ , which makes the latter more preferable for applications.

In quantum communication of continuous variables the two-mode squeezed vacuum plays a central role. We have applied the Peres-Horodecki separability criterion for continuous variable systems to the state obtained after transmission of a squeezed vacuum through two absorbing fibers at finite temperature. The result reveals that there is a critical length at which the initially entangled state becomes separable. Clearly, the critical length can tell us only whether or not

the transmitted light contains entanglement. It cannot not, however, tell us anything about the amount of entanglement that is preserved, i.e., the scale of entanglement degradation. This is a rather complicated problem, because of the drastically increasing dimension of the Hilbert space with increasing squeezing strength, and will be considered elsewhere.

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