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RENORMALIZATION GROUP SYMMETRIES FOR SOLUTIONS OF BOUNDARY VALUE PROBLEMS[1]

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A progress in constructing renormalization group symmetries by means of a regular approach is described. A basic sketch of general ideas of the algorithm is followed by several illustrations for solutions of boundary value problems in plasma physics.

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1 Introduction

A method for improving approximate solutions that take advantage of the symmetry of the particular solution, the so-called Renormalization Group Method (RGM), is known in theoretical physics since mid-fifties [1]. The essence of this method as used in Quantum Field Theory (QFT) [2] lies in the group scaling transformation of an independent variable (and, possibly some parameter) accompanied by the functional transformation of some solution characteristic. These transformations are used for a successive improvement of a set of approximate solutions that are expressed in the form of the series in powers of some small parameter. Advances of RGM in QFT is due to some functional equation that guarantees the existence of the solution group property and is supposed either to be known or borrowed from some additional physical assumption.

The speedy proliferation of RGM ideas from QFT to other fields of theoretical physics [3] is owing to the common to various physical systems the property of Functional Self-similarity (FS) [4], that makes it possible to apply RGM to different physical models. At the same time for boundary value problems (b.v.p.) that are based on differential and/or integro-differential equations a new algorithm was elaborated [5]. This algorithm appeared in some sense close to RGM in QFT yet it exploited a different, more adequate to mathematical models employed, method of constructing the solution symmetries. In fact thanks to the new algorithm a notion of Renormalization Group Symmetry (RGS) came into being. There were two reasons for calling them Renormalization Group (RG) Symmetries: firstly, from mathematical point of view the calculational algorithm for these symmetries is very similar to that used in modern group analysis

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and, secondly, they are applied to perturbative theory (PT) solutions with the goal usual to RGM, i.e. to improve these PT solutions. It was due to utilization of modern group analysis technique that made it possible the appearance of the regular algorithm of RGS constructing. In this report I will briefly touch upon the main features, which distinguish this approach from the already known in theoretical physics, and illustrate this difference by several examples of b.v.p. solutions in plasma physics.

2 Principal stages in evolution of the algorithm of RGS constructing

The milestones in developing the algorithm of RGS constructing were reported successively on all RG meetings since 1991 and are shortly listed below:

RG-'91 ⇒ Preliminary results in applying group analysis methods for RGS constructing
- RGS constructing for initial value problem for Burgers equation,
- Utilization of RGS for constructing the plasma permittivity tensor,
- RGS constructing using embedding equations for ordinary differential equations,
- RGS in the problem of the harmonics generation in inhomogeneous plasma

RG-'96 ⇒ RGS constructing scheme and illustrations for b.v.p. in nonlinear optics
- Lie-Bäcklund RGS and exact solutions in nonlinear geometrical optics
- Approximate RGS for the beam self-focusing problem in nonlinear geometrical optics

RG-'99 ⇒ Modern form of the scheme for the RGS constructing and applications and its relations to other algorithms
- Approximate and exact RGS for the wave beam self-focusing problem, approximate RGS in two parameters, two-dimensional solution singularity
- Multi-parameter RGS

RG-2002 ⇒ RGS algorithm for physical systems based on nonlocal equations
- Exact and approximate RGS for b.v.p., that are described by integro-differential equations
- RGS for the problem of plasma bunch expansion

The scheme for constructing RGS was already discussed on RG conferences, and particularly in details on the preceding RG-99 [6]. So here I will only briefly recall the main steps of this scheme in order to distinguish it from the existing RG methods. Conventionally the scheme for constructing RGS, depicted on Fig. 1 below, is realized as a sequence of four steps that are:

1) constructing the specific RG manifold

2) calculating the symmetry group that leaves this manifold invariant

3) restricting this manifold to a particular (usually approximate) b.v.p. solution with the goal to obtain the desired RGS

4) utilizing this RGS to obtain the particular b.v.p. solution

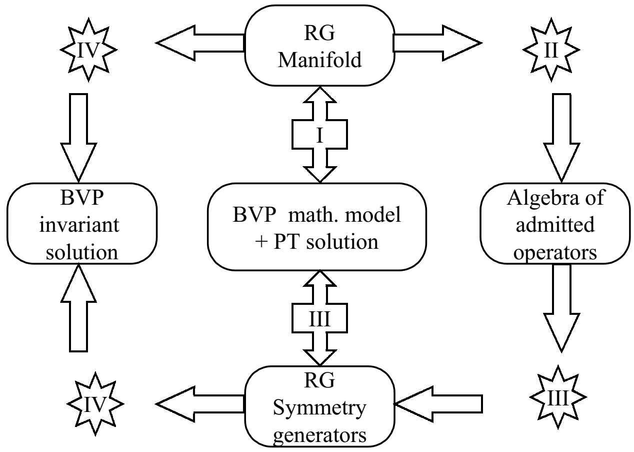

Fig. 1. The general scheme for RGS constructing.

Let us outline the key points for these steps.

For the first step **(I)** it is the specific way of involving in group transformations parameters and/or boundary conditions, that define the particular solution of interest. We suppose that we know the approximate representation for this solution in the form of a truncated power series either in small parameter related to boundary (or initial) conditions or in small deviation from the region boundary where the solution is given. The success of the first step depends on many factors, namely a) the possibility of expressing boundary conditions as embedding equations or as additional differential constraints, b) the possibility of introducing perturbation theory parameters in basic equations, c) the chance to treat the solution in the extended space of independent and dependent variables, parameters and differential variables, d) the eventuality to make use of small parameters that differ from that used in PT solution [7]. The particular form in which the first step, i.e. constructing RG manifold, is realized is mainly dependent on the form of mathematical model employed and the form of representation of the approximate PT solution.

The second step **(II)** of the scheme employs the standard techniques of modern group analysis. It is based on regular mathematical methods and uses infinitesimal approach for calculating symmetry group generators. Here we do not restrict ourselves to "scaling-type" transformations only, but also allow multi-parameter groups and, moreover, these should not be necessary point groups. Hence here we extend the notion of RG operator and FS, which traditionally was based only on one-parameter point groups.

The distinctive feature of the third step **(III)** is that it demonstrates the difference of the described approach both from the classical Bogoliubov RGM used in QFT and the standard group analysis. Indeed, the classical group analysis is typically employed to calculate the general transformation group for the system of equations "as a whole", without specifying the form of boundary conditions. The group obtained is then used to find optimal system of subalgebras and construct the related invariant and partial invariant solutions with boundary solutions that are obtained while treating every particular solution. On the other hand the impossibility to deduce the exact expressions for β-functions in Bogoliubov RG (see, however, the “Exact RG” constructing approach [8]), that usually is defined by truncated PT series, is the main obstacle that makes the RG approach approximate just by its nature.

On the contrary, the procedure of checking invariance condition fulfilled on the third step **(III)** for the group generators with the predefined (on the second step **(II)**) coordinates (or β-functions in terms of QFT) and in view of the particular b.v.p. solution in the form of truncated PT series is not subjected to these drawbacks. This procedure solves two problems: firstly, it gives the specific transformation group that just by the method of its construction leaves the solution of the b.v.p. with specified boundary conditions invariant. Hence it eliminates the necessity to investigate all possible invariant solutions with the goal to find the solution with the desired boundary condition. Secondly, it allows to point the exact (in the framework of the group analysis scheme) form of the generator that guarantees the transformation of the approximate PT solution to the exact solution of the b.v.p.

The fourth, last step **(IV)** looks standard, i.e. it is in the root of a usual procedure of finding invariant solution for a given set of group generators and has been detailed in various monographs [9–11].

To distinguish the described approach from mathematical structures employed in theoretical physics we note that more likely is the common terminological nature of the notions used, particularly to QFT methods, where the use of a solution symmetry serves as the instrument for improving perturbative solutions, but the approaches to finding these symmetries are different. So there is no direct analogy neither to Wilson RG construction used in critical phenomena [12], nor to approaches, which are applied in the theory of turbulence [13].

To conclude the discussion of the RGS scheme we note that different forms of the RG implementation, used in theoretical and mathematical physics, namely:

a) Bogoliubov RG in QFT and some other fields of macroscopic physics, where RG symmetry appears as the exact solution symmetry formulated in terms of intrinsic variables;
b) RG in turbulence and continuous spin-field models, where RGS is the symmetry of some auxiliary QFT model;
c) Wilson RG in phase transition theory, theory of polymers and percolation, which is based on Kadanoff-Wilson blocking procedure, and where RG transformation is a transformation between different auxiliary (specially constructed) models of a given system

is supplemented by one additional type of RG symmetry,

d) RGS for b.v.p. in mathematical physics. Similar to QFT it reflects the invariance property of the solution with respect to group transformations involving both intrinsic variables, boundary conditions and parameters though this RGS is constructed in a regular way using the described algorithm.

This algorithm can be applied to any mathematical model that is based on differential or integro-differential equations, so I will illustrate the advantages of the method for plasma theory. A large variety of nonlinear processes in plasma physics are described in terms of Vlasov-

Maxwell equations, i.e. kinetic equations for plasma particles in a self-consistent electromagnetic field governed by Maxwell equations. This mathematical model is used, for example, to describe nonlinear propagation and self-focusing of a powerful wave beam in plasma, harmonics generation process, acceleration of ions in plasma expansion process and so on.

Example I. Harmonics generation process in inhomogeneous plasma.

This example appears to be the first demonstrations of the advantageous use of RG algorithm for nonlinear physics. The physical problem here is formulated as follows: an inhomogeneous plasma with a smoothly varying density profile is irradiated by a powerful laser which produces the incident p-polarized electromagnetic wave. Due to the effect of the linear transformation this wave creates a strong potential electric field in the vicinity of the plasma resonance where the wave frequency, ω, coincides with the local electron Langmuir frequency, ω_{Le}. The potential field generates higher harmonics $n\omega$, that are then radiated from plasma. The portion of the laser incident energy which is re-radiated from plasma via harmonics is defined by laser flux q, the angle of incidence θ of the p-polarized wave and plasma parameters, namely the temperature T and the characteristic density inhomogeneity scale L in the vicinity of plasma resonance. Usually the problem of calculating the harmonics conversion coefficient in solved in the weak nonlinearity limit, hence the influence of higher harmonics on lower ones is neglected. The increase of the laser flux density q violates the weak nonlinearity approximation and one is forced to take into account effects of higher harmonics. The RG algorithm appears to be powerful enough to meet the challenge.

Let us point to some specific problems that are encountered here while constructing RGS. In case of the weak electron thermal motion and fixed ions a more simple (as compared to kinetic one) hydrodynamic model is used to describe electron dynamics. For small values of the angle of incidence $\theta \ll 1$ and low plasma density gradients ($L \to \infty$) the influence of the plasma nonlinearity is mainly in the vicinity of plasma resonance, hence instead of treating the general system of *six* plasma collisionless hydrodynamics equations (for three components of electromagnetic fields, two components of electron velocity and their density) it appears possible to use as RG manifold only *two* equations (for components of the electron velocity and the electric field along the density gradient). The procedure of restriction of the infinite dimensional group admitted by this manifold to the approximate solution which is obtained in linear approximation with respect to the basic system of *six* equations gives the desired RGS with the group parameter, $\propto \sqrt{q}$, proportional to the amplitude of the incident electromagnetic wave. The electric field and the electron velocity are invariants of group transformations while the transformation of the coordinate is related to the structure of the potential electric field near plasma resonance in the linear approximation.

This example is the distinct illustration of the advantageous use of the approach based on RG symmetries. First, it is the universality, which manifests itself in the fact that both RGS and the b.v.p. solution, respectively, can be constructed for various initial conditions that define the electric field structure near plasma resonance. Second, one can use in the restriction procedure the PT solution obtained from a more complicated equations as compared to RG manifold. Third, it is the possibility to improve both RGS and the solution obtained by taking into account the omitted terms proportional to small parameters such as density gradient and so on.

The main physical results here that gives the RG algorithm is analytical formulas for the nonlinear structure of the electric field near the plasma resonance that enables to calculate harmonics spectrum both localized in plasma and radiated in vacuum. The typical analytical dependence of of the harmonics conversion coefficient K_n has the form [14]:

$$K_n \sim q^{n-1} \mid \mathcal{K}_1 + (q/q_0)\mathcal{K}_2 \mid^2 , \quad n \geqslant 2 ,$$

where complex functions $\mathcal{K}_i$ depend upon the harmonics number n, the plasma resonance width Δ and the angle of incidence θ. For $q/q_0 \to 0$, that corresponds to the weak nonlinearity limit, K_n is defined only by the first term $\sim \mathcal{K}_1$ and correlates with the expression, obtained earlier in perturbative approximation. For strong nonlinearity, when the density flux q is comparable to the breakdown flux q_0 for electron Langmuir oscillations $(q_0 \geqslant q)$, nonlinear effects yield a dependence of K_n upon n, Δ and θ, which differ from that given by the weak nonlinearity theory. Another significant strong nonlinear effect is the new set of oscillations of K_n, which appear with the rise in the laser plasma temperature T [15].

Example 2. Nonlinear plasma permittivity and a three-dimensional RGS.

The plasma nonlinear permittivity (NPP) defines the nonlinear current density – the nonlinear material equation – of the corresponding order $l > 1$ in powers of the electric field. It is of interest for investigating nonlinear processes that are treated in terms of the integro-power dependence of the current density upon the self-consistent electric field, for example, wave-particles scattering, parametric instabilities, harmonics generation. Usually, NPP for hot plasma is obtained by iterating Vlasov kinetic equation with respect to self-consistent field, whilst the NPP for cold plasma follow from the collisionless plasma hydrodynamic equations. For high order nonlinearity $(l \geqslant 4)$ the procedure of NPP tensor symmetrization in hot plasma appears more tedious in hot plasma than in cold one. The utilization of the RG approach here establish a one-to-one correspondence of NPP tensors in hot and cold plasmas for arbitrary order of nonlinearity and gives an algorithm that enables to restore "hot" tensors from the corresponding "cold" formulas.

The classical Lie algorithm should be modified in this example [16] because RG manifold is given by equations with nonlocal terms, and this is the first peculiarity of RGS constructing here. The second is that NPP tensors are defined in $(\omega, \boldsymbol{k})$-representation and we need to define the procedure of extension of group generators to Fourier variables [17]. The third peculiarity is that the RGS obtained is a vector (three-dimensional) group. For example, in non-relativistic plasma the corresponding RG operator in the space of Fourier variables $\{\omega, \boldsymbol{k}, \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}}, \hat{\rho}, \hat{\boldsymbol{j}}\}$ (denoted by "hats") results from the Galilean group operator

$$\mathbf{R} = \boldsymbol{k}\partial_\omega + \partial_{\boldsymbol{v}} - (1/c)\left[\hat{\boldsymbol{B}},\, \partial_{\hat{\boldsymbol{E}}}\right] + \hat{\rho}\partial_{\hat{\boldsymbol{j}}} \,.$$

Here the group parameter $\boldsymbol{v}$ defines the "velocity" of a group of moving particles. The utilization of the finite transformations in the relation that defines "partial" current density for the given group of particles and the subsequent "averaging", i.e. integrating over group parameter with the "weight" function proportional to the equilibrium particle distribution function $f_0(\boldsymbol{p})$ gives the desired NPP tensor in hot plasma

$$\varepsilon_{ij_1\ldots j_n}(\omega_1, \boldsymbol{k}_1; \ldots; \omega_n, \boldsymbol{k}_n) = \frac{1}{\omega\omega_1\ldots\omega_n}\int \mathrm{d}\boldsymbol{p} f_0(\boldsymbol{p})(\omega - \boldsymbol{k}\boldsymbol{v})(\omega_1 - \boldsymbol{k}_1\boldsymbol{v})\ldots(\omega_n - \boldsymbol{k}_n\boldsymbol{v})$$

$$\bar{\varepsilon}_{ab_1\ldots b_n}(\omega_1 - \boldsymbol{k}_1\boldsymbol{v}, \boldsymbol{k}_1; \ldots; \omega_n - \boldsymbol{k}_n\boldsymbol{v}, \boldsymbol{k}_n)\beta_{ai}(\omega, \boldsymbol{k})\beta_{b_1 j_1}(\omega_1, \boldsymbol{k}_1)\beta_{b_n j_n}(\omega_n, \boldsymbol{k}_n)\,; \quad n \geqslant 2;$$

$$\omega = \omega_1 + \ldots + \omega_n\,;\ \boldsymbol{k} = \boldsymbol{k}_1 + \ldots + \boldsymbol{k}_n\,;\ \int \mathrm{d}\boldsymbol{p} f_0(\boldsymbol{p}) = 1\,, \quad \beta_{ij}(\omega, \boldsymbol{k}) = \delta_{ij} + \frac{k_i v_j}{\omega - \boldsymbol{k}\boldsymbol{v}}\,.$$

Here $\bar{\varepsilon}$ is related to NPP tensor of the cold collisionless plasma without external fields. For example, for $n=2$ it is given by the formula

$$\bar{\varepsilon}_{isj}(\omega_1 - \boldsymbol{k}_1\boldsymbol{v}, \boldsymbol{k}_1; \omega_2 - \boldsymbol{k}_2\boldsymbol{v}, \boldsymbol{k}_2) = -\frac{4\pi i e^3 n_e}{2! m^2}\left[(\omega - \boldsymbol{k}\boldsymbol{v})(\omega_1 - \boldsymbol{k}_1\boldsymbol{v})(\omega_2 - \boldsymbol{k}_2\boldsymbol{v})\right]^{-1} \times \\ \times \left(\frac{k_i}{\omega - \boldsymbol{k}\boldsymbol{v}}\,\delta_{js} + \frac{k_{1s}}{\omega_1 - \boldsymbol{k}_1\boldsymbol{v}}\,\delta_{ij} + \frac{k_{2j}}{\omega_2 - \boldsymbol{k}_2\boldsymbol{v}}\,\delta_{is}\right),$$

where e and m denote the charge and mass of plasma electrons with the density n_e.

A similar result for the relativistic plasma can be obtained using as a starting point not the Galilean transformation group but the group of Lorentz transformations.

Example 3. Acceleration of ions and dynamics of the plasma bunch expansion.

This example provides yet another illustration of RG algorithm application to physical systems based on nonlocal equations. Interest in this problem stems primarily from the need to better understand the physics of ion acceleration in the interaction of laser light with plasma and in particular to give a quantitative description of the ion acceleration. The study of ion acceleration is among the key problems in various applications of high power lasers, such as laser fusion, injectors of fast particles, and radioactive sources for apparatuses used in medicine and nuclear physics. Recent experiments with short-lived (nanosecond) plasmas [18, 19] and thin foils [20] confirmed the existence of a special expansion regime for small plasma bunches which is essentially unsteady and is accompanied by the adiabatic cooling of plasma particles.

The investigation of this expansion regime using a phenomenological hydrodynamic theory and numerical modeling recently was supplemented by the exact solution of kinetic equations [21]. However, this solution was obtained in a particular case of quadratic dependence of electrostatic potential in spatial coordinate and, accordingly, this imply the same dependence of plasma particles distribution functions on the coordinate and velocity thus imposing strict limitations on the possible application of these results to analyze experimental data. The utilization of RGS allows to make the next step in analytical investigation of plasma particles dynamics in bunch expansion process [22].

Mathematically this problem in one-dimensional (plane) geometry is formulated as the initial value problem for the system of kinetic equations for particles distribution functions f^α of every sort α with the given initial values f_0^α,

$$\partial_t f^\alpha + v\partial_x f^\alpha + (e_\alpha/m_\alpha)E\partial_v f^\alpha = 0\,, \quad f^\alpha\Big|_{t=0} = f_0^\alpha(x,v)\,.$$

Here E is the self-consistent electric field, that obey the Poisson equation in which the charge density is defined by zero-order moments of the particles distribution functions.

For plasma flows with characteristic scale length for plasma-density variations large compared with a Debye length we use quasi-neutral approximation and set the charge and the current densities, ρ and j, equal to zero,

$$\rho \equiv \int \mathrm{d}v\,\sum_\alpha e_\alpha f^\alpha = 0\,, \quad j \equiv \int \mathrm{d}v\,v\,\sum_\alpha e_\alpha f^\alpha = 0\,.$$

The electric field E is then expressed in terms of the moments of the distribution functions. This simplification of the mathematical model leads to the RG manifold that admits the transformation

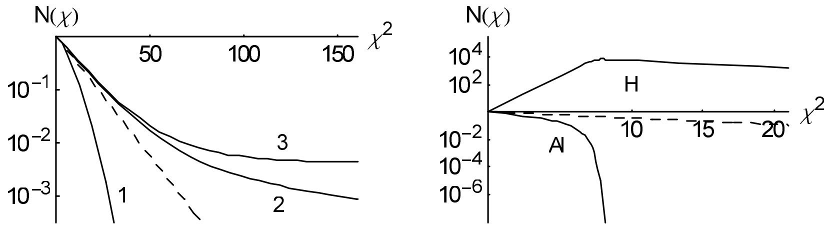

Fig. 2. Ion density distributions in (e,i) plasma (left) and in plasma with protons and Al-ions (right).

group extended (as compared to [23]) by one additional conformal group operator. The restriction procedure of the group obtained to the approximate solution of the initial value problem at $t \to 0$ gives the desired RG operator that allows to extend this solution to large values of t and obtain a more general class of solutions to the Cauchy problem for the Vlasov equations in quasi-neutral approximation. These solutions are valid for arbitrary initial distribution functions of particles, say, for two-temperature Maxwellian and super-Gaussian initial electron distributions and Maxwellian ions with different temperatures, densities and masses of ion species. Along with the plasma particles distribution functions RG approach also gives analytical expressions for integral characteristics, namely local ion density and ion energy distribution function, that may be used for analyzing experimental data.

To illustrate the aforesaid on Fig. 2 we present a sketch of the typical ion density distributions, $n_q = (n_{q0}/\sqrt{1+\Omega^2 t^2})\, N_q(\chi)$ for plasma with one and two ion species. Here $\chi = \Omega x/\sqrt{1+\Omega^2 t^2} V_{Ti}$, V_{Ti} is the proton thermal velocity and the frequency Ω is defined by the ratio of the acoustic speed to the initial size of plasma bunch. The index q points to ions ($q \equiv i$) in (e,i) plasma and $q = \{Al, H\}$ in plasma with two ion species.

The left figure gives the ion density distribution in the two-component (e,i) plasma with different initial electron distribution functions and for Maxwellian ion initial distribution function. Curve 1 corresponds to super-Gaussian electron distribution, and curves 2 and 3 are related to two-temperature Maxwellian distribution with hot to cold temperature ratio $T_h/T_{e0} = 10$ (2), 100 (3) and the relative density $n_{h0}/n_{i0} = 0.1$. The dashed curve corresponds to Maxwellian initial distribution function for electrons. One can see that the presence of hot electrons leads to enriching of the ion energy spectrum by high-energy ions (the plots $N(\chi)$ after the substitution χ by v/V_{Ti} characterize the ion energy spectrum distribution at large $t \gg 1/\Omega$).

The right figure presents ion density distributions for plasma with two ion species, namely heavy Al-ions and protons, which form the low-density impurity with the maximum initial relative density $n_{H0}/n_{Al0} \approx 0,00077$. It is obvious that even a small amount of light ions impurity substantially reduces the acceleration of the general body of heavy ions. The results obtained show promise of clarifying the nature of the ion high-energy spectrum cut-off [18, 19] and to predict the energy of a given sort of ions in multi-component plasma.

If we do not use the quasi-neutral approximation the initial value problem for plasma bunch dynamics is solved using the approximate RGS with the electric field amplitude treated as a small parameter. Here, unlike the quasi-neutral limit the RG manifold is given by the full system of Vlasov-Maxwell equations with the material relations that define the charge and current densities for every sort of particles. The approximate RGS obtained describes the evolution

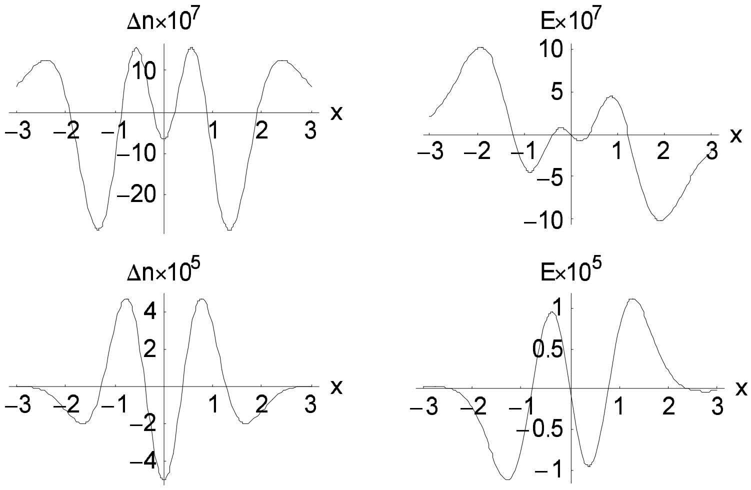

Fig. 3. Plasma densities disturbances Δn (in units of the maximum density n_{max}) and the electric field E (in units of $2\pi e n_{max} d$) versus coordinate x, normalized by the characteristic initial size d of a plasma bunch.

of particles distribution functions and the electric field on the initial stage of plasma dynamics when nonlinear oscillations of plasma density are excited. A sketch of the typical plasma densities disturbances $\Delta n = n_e - Zn_i$ and the related electrostatic field E space distributions are presented on Fig. 3 for (e, i) plasma at different moments of time: $t = 4/\omega_{Le}$ for the upper panel and $t = 6/\omega_{Le}$ for the lower panel. The initial electron and ion distribution functions are Maxwellian with $T_{e0}/T_{i0} = 10$ and the initial density space distribution is described by the Gaussian curve.

3 Conclusion

What are the feasible future trends in evolution of the RG algorithm? First, we may draw our attention to the problems that have been already investigated by the traditional RG method, say, in QFT, and try to reformulate them in terms of the new RG algorithm that may give an additional impulse in overcoming the encountered difficulties. Second, the development of RG approach will provide a better understanding of the physical meaning of new possible types of the RGS that lead not to invariant but to partially-invariant solutions, that have not been discussed in RG application so far. Third, we shall continue developing the algorithm of RGS constructing based on methods of modern group analysis and enlarge the number of b.v.p. that can be analyzed by this algorithm. Fourth, the need for RGS in various physical problems may bring to the forefront new mathematical objects to study with the methods of modern group analysis. For example, the problem of constructing RGS in the space of distributions is of particular interest.

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