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EXAMPLE OF A RENORMALIZATION GROUP SOLUTION FOR POINCARÉ TRANSFORMATION IN THE FOCK SPACE[1]

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The whole set of correctly commuting renormalized Poincaré generators is presented up to the terms of second order in the coupling constant in the case of $g\phi^3$ theory. It is explained how Poincaré group elements are obtained by the exponentiation of generators in perturbation theory. A dynamical Lorentz transformation of one-particle eigenstate of the effective Hamiltonian is shown as an example.

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1 Introduction

Every quantum field theory of interest leads to divergences caused by direct coupling of particle states with small and large invariant masses because of interaction terms present in the generators of Poincaré algebra. One could try to regularize the generators, but this introduces a new artificial parameter to the theory and violates commutation relations in the Poincaré algebra. The similarity renormalization group procedure for particles [1,2] provides a method for constructing a well defined Hamiltonian whose eigenstates are independent of regularization. The procedure can be extended to all other generators of Poincaré group [3, 4]. This presentation is showing it in practice using the simplest possible interaction, $g\phi^3$.

2 Effective Poincaré algebra

Canonical expressions for ten generators of the Poincaré group P^μ and $M^{\mu\nu}$ [5] have to be regularized to make theory finite. Every dynamical generator A_∞ in the $g\phi^3$ theory has the form

$$A_\infty = \int [p] v_\infty^{(0)}(p) a_p^\dagger a_p + g \int [p_1 p_2 p_3] \delta(p_1 + p_2 - p_3)\, r_\Delta\, v_\infty^{(1)}\; a_1^\dagger a_2^\dagger a_3 + H.c. \,, \tag{1}$$

where $[p]$ is the measure in momentum space, r_Δ is regularization function introduced to make these operators finite and v's are given functions of momenta or differential operators acting in

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the momentum space. Creation and annihilation operators are sometimes labeled by $1,\ 2,\ \ldots$ instead of $p_1,\ p_2, \ldots$ in this article in order to simplify the notation.

The regularization violates Poincaré algebra commutation relations and therefore the bare regularized generators cannot be accepted.

With the help of similarity transformation the annihilation and creation operators corresponding to the scale $\lambda = \infty$, i.e. the annihilation and creation operators in any bare generators A_∞, are expressed in terms of effective ones, corresponding to the scale $\lambda = \infty$.

$$a^\dagger_\lambda = U^\dagger_\lambda a^\dagger_\infty U_\lambda \ . \tag{2}$$

Here, U_λ is unitary transformation which is a solution of the renormalization group equation [1]. Up to the second order in g this gives

$$\begin{aligned} a^\dagger_{\infty p} &= [1 + g^2 h(\lambda, \Delta)] a^\dagger_{\lambda p} \\ &+ g \int [p_1 p_2] \Big[\delta(p_1 + p_2 - p)\ z_{1\lambda}\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} + \delta(p_2 - p_1 - p)\ z_{2\lambda}\ a^\dagger_{\lambda 2} a_{\lambda 1} \Big] \\ &+ g^2 \int [p_1 p_2 p_3] \Big[\delta(p_1 + p_2 + p_3 - p)\ z_{3\lambda}\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} a^\dagger_{\lambda 3} \\ &+ \delta(p_1 + p_2 - p_3 - p)\ z_{4\lambda}\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} a_{\lambda 3} + \delta(p_3 - p_1 - p_2 - p)\ z_{5\lambda}\ a^\dagger_{\lambda 3} a_{\lambda 1} a_{\lambda 2} \Big] \ , \end{aligned} \tag{3}$$

where h is a specified function and z's are functions of momenta of particles.

Inserting the expression for $a^\dagger_{\infty p}$ from eq. (3) into eq. (1) and keeping terms up to the second order in g, one can find the form of needed counterterms in the generators. The counterterms are found from the condition that the matrix elements of effective generators A_λ between states with small invariant masses are independent of Δ in the $\Delta \to \infty$ limit. Thus the effective generators A_λ are finite.

$$\begin{aligned} A_\lambda &= \int [p] [v^{(0)}_\lambda(p) + g^2 c_\lambda(p)] a^\dagger_{\lambda p} a_{\lambda p} \\ &+ g \int [p_1 p_2 p_3] \delta(p_1 + p_2 - p_3) f_\lambda\ v^{(1)}_\lambda\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} a_{\lambda 3} \\ &+ g^2 \int [p_1 p_2 p_3 p_4] \delta(p_1 + p_2 + p_3 - p_4) f_\lambda\ v^{(2)}_{1\lambda}\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} a^\dagger_{\lambda 3} a_{\lambda 4} \\ &+ g^2 \int [p_1 p_2 p_3 p_4] \delta(p_1 + p_2 - p_3 - p_4) f_\lambda\ v^{(2)}_{2\lambda}\ a^\dagger_{\lambda 1} a^\dagger_{\lambda 2} a_{\lambda 3} a_{\lambda 4} + H.c.\ . \end{aligned} \tag{4}$$

Here, v's are functions of momenta of particles. The effective dynamical generators get the similarity form factors f_λ which limit possible momenta transfer in the vertex. The function f_λ may be chosen for example as

$$f_\lambda = \exp\left(-\frac{(M^2_{\rm ann} - M^2_{\rm cre})^2}{\lambda^4} \right) \ , \tag{5}$$

where $M^2_{\rm ann}$ and $M^2_{\rm cre}$ are the squares of invariant masses of annihilated and created particles. The kinetic term in eq. (4) is corrected by second order term $g^2 c_\lambda(p)$ coming from the loop integration. It represents a correction to the mass of a particle. The first order terms are changed only by the presence of similarity form factor f_λ, i.e. $v_\lambda^{(1)} = v_\infty^{(1)}$. All second order terms are new, produced by similarity transformation and they have similar form factors.

After similarity transformation all kinematical generators preserve their form, only creation and annihilation operators therein correspond now to the scale λ, i.e. $a_{\infty p}^\dagger$ and $a_{\infty p}$ are replaced by $a_{\lambda p}^\dagger$ and $a_{\lambda p}$, respectively.

The presence of the similarity form factor f_λ in the effective dynamical generators allows us to remove regularization functions r_Δ from the expression for A_λ. It can be checked by an explicit calculation that effective generators P_λ^μ and $M_\lambda^{\mu\nu}$ satisfy all required commutation relations in the Poincaré algebra.

3 Dynamical transformation of a physical state

Any Poincaré transformation is obtained by exponentiation of generators of Poincaré algebra.

$$|\Psi_\alpha(\vec{p})\rangle = L(\alpha)|\Psi(\vec{p})\rangle \ , \tag{6}$$

where

$$L(\alpha) = e^{-i\alpha A_\lambda} \ . \tag{7}$$

In the case of a free theory this yields to

$$|\Psi_\alpha(\vec{p})\rangle = |\Psi(\vec{p}_\alpha)\rangle \ , \tag{8}$$

where $p_\alpha = \Lambda(p_\alpha \leftarrow p)\, p$. Here, $\Lambda(p_\alpha \leftarrow p)$ is the Lorentz transformation corresponding to $L(\alpha)$ and $p = (\vec{p}, E)$, $p_\alpha = (\vec{p}_\alpha, E_\alpha)$ and $p^2 = p_\alpha^2 = m^2$.

When the interaction is present, the explicit calculation of the exponent in eq. (7) has to deal with divergences of the local quantum field theory. Therefore, the calculation of $L(\alpha)$ is possible only with a correctly defined operator A_λ such as the generators of the renormalized Poincaré algebra are.

Now, $L(\alpha)$ is calculated similarly to the S-matrix in the old fashioned perturbation theory. One can write

$$A_\lambda = A_{\lambda 0} + A_{\lambda I} \ , \tag{9}$$

where $A_{\lambda 0} = A_\lambda(g = 0)$ and introduce $W(\alpha)$ such that

$$L(\alpha) = W(\alpha) L_0(\alpha) \ , \tag{10}$$

where $L_0(\alpha) = \exp(-i\alpha A_{\lambda 0})$. Here, $W(\alpha)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} W(\alpha) = -iW(\alpha) A_{\lambda I}(\alpha) \ , \tag{11}$$

where $A_{\lambda I}(\alpha) = L_0(\alpha) A_{\lambda I} L_0(\alpha)^{-1}$, which may be integrated term by term in a power series in g. One obtains

$$W(\alpha) = 1 + gW_1(\alpha) + g^2 W_2(\alpha) + \cdots \ , \tag{12}$$

where

$$W_1(\alpha) = -i\int_0^\alpha \mathrm{d}\beta A_{\lambda 1}(\beta)\ , \tag{13}$$

$$W_2(\alpha) = (-i)^2\int_0^\alpha \mathrm{d}\beta\int_0^\beta \mathrm{d}\beta' A_{\lambda 1}(\beta')A_{\lambda 1}(\beta) - i\int_0^\alpha \mathrm{d}\beta A_{\lambda 2}(\beta)\ . \tag{14}$$

In this way the transformation may be found order by order.

In the example discussed in reference [3], a rotation around x axis is analyzed, because in the light front formulation of dynamics the rotation is dynamical [6]. In the equal time formulation the boosts are dynamical.

The check that effective algebra has to pass is: does it properly transform physical states?

First we define a state which will be transformed. The eigenstate equation for one-particle state

$$H_\lambda|\Psi_\lambda(\vec{p})\rangle = E_{\mathrm{phys}}|\Psi_\lambda(\vec{p})\rangle \tag{15}$$

is solved in perturbation theory up to second order in g. Note that eigenvalue E_{phys} is independent of λ though both H_λ and $|\Psi_\lambda(\vec{p})\rangle$ are dependent on. The mass of the physical particle is connected to the bare mass by

$$m^2_{\mathrm{phys.}} = m^2 + g^2\delta m^2\ . \tag{16}$$

The term $g^2\delta m^2$ is the one loop correction to the bare mass and it comes from a solution to the renormalization group equation. We introduce $p_{\mathrm{phys}} = (\vec{p}, E_{\mathrm{phys}})$ for which $p^2_{\mathrm{phys}} = m^2_{\mathrm{phys}}$ in contrast to $p^2 = m^2$. Up to the second order in g, the one particle eigenstate is

$$|\Psi(\vec{p})\rangle = N\left[|\Psi_0(\vec{p})\rangle\ + g|\Psi_1(\vec{p})\rangle + g^2|\Psi_2(\vec{p})\rangle + \cdots\right]\ , \tag{17}$$

where N is a normalization factor and $|\Psi_0(\vec{p})\rangle = a^\dagger_{\lambda p}|0\rangle$. The states $|\Psi_1(\vec{p})\rangle$ and $|\Psi_2(\vec{p})\rangle$ are calculated using perturbation theory.

According to eq. (6), the transformed state is

$$\begin{aligned} |\Psi_\alpha(\vec{p})\rangle &= N\{L_0(\alpha)|\Psi_0(\vec{p})\rangle + g\left[L_1(\alpha)|\Psi_0(\vec{p})\rangle + L_0(\alpha)|\Psi_1(\vec{p})\rangle\right] \\ &+ g^2\left[L_2(\alpha)|\Psi_0(\vec{p})\rangle + L_1(\alpha)|\Psi_1(\vec{p})\rangle + L_0(\alpha)|\Psi_2(\vec{p})\rangle\right]\} \\ &= N\left[|\Psi_0(\vec{p}_\alpha)\rangle + g|\Psi_1(\vec{p}_\alpha)\rangle + g^2|\Psi_2(\vec{p}_\alpha)\rangle + g^2\delta m^2\frac{\mathrm{d}}{\mathrm{d}m}|\Psi_0(\vec{p}_\alpha)\rangle\right]\ . \end{aligned} \tag{18}$$

All terms but the last one are what one could expect having experience from a free field case. The last term in eq. (18) can be understood as follows. The physical state is labeled by $\vec{p}$ and it is the eigenstate of H_λ with the eigenvalue E_{phys}. Further on, $\vec{p}$ and E_{phys} form a four-vector p_{phys}. The Poincaré transformation we consider here changes p_{phys} to $p_{\mathrm{phys}\alpha} = (\vec{p}_{\mathrm{phys}\alpha}, E_{\mathrm{phys}\alpha})$, thus the transformed state should be $|\Psi(\vec{p}_{\mathrm{phys}\alpha})\rangle$. Because the considered transformation is dynamical, then the definition of $\vec{p}_{\mathrm{phys}\alpha}$, which should label the transformed state, involves the physical mass m_{phys}. Therefore, if we perform the calculation up to the second order in g, we have to expand $|\Psi(\vec{p}_{\mathrm{phys}\alpha})\rangle$ in the vicinity of $|\Psi(\vec{p}_\alpha)\rangle$. We have to do this only for the

$|\Psi_0(\vec{p}_{\mathrm{phys}\alpha})\rangle$ component because the physical and bare masses differ by the term of order g^2. Thus

$$a^\dagger_{\lambda p_{\mathrm{phys}\alpha}} = a^\dagger_{\lambda p_\alpha} + g^2 \delta m^2 \frac{\mathrm{d}}{\mathrm{d}m} a^\dagger_{\lambda p_\alpha} \tag{19}$$

shows the source of the last term in eq. (18). Details of the calculation can be found in [3].

4 Conclusion

The renormalized Poincaré group generators obtained for $g\phi^3$ satisfy all required commutational relations up to the order g^2 including the hard-to-satisfy relation between dynamical generators. Exponentiation of the generators gives a representation of Poincaré group that correctly transforms physical states as it is required by special relativity.

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