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RELATIVISTIC PERIASTRON SHIFT OF A PARTICLE IN KERR FIELD: A PARTICULAR CASE-STUDY

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This letter presents a calculation of periastron shift in an astrophysical binary in which a non-spinning body moves in a slightly deformed circle round a slowly spinning central body. The equation of motion followed in the calculation was derived earlier by Faruque using relativistic effective one-body dynamics. The results show a clear dependence of periastron shift on magnitude as well as on direction of spin and orbital angular momentum of the central body and the orbiter, respectively, and this dependence is in accord with previous results found by a different procedure by other authors.

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In an astrophysical binary system the periastron shift is due to Newtonian and relativistic contributions. The Newtonian effects on the periastron shift are well known. A detailed discussion of this can be found in [1]. Regarding the relativistic effects on periastron shift we know that the dominant contribution was found by Einstein[2] in the calculation of Mercury's relativistic perihelion shift. In strong gravitational fields, i.e., near binary pulsars or black holes, higher order relativistic contributions due to spin-orbit and spin-spin interactions could be relevant. Damour and Schafer [3] found an exact solution useful to calculate, in the test-particle approximation, the relativistic periastron shift in binaries in which one of the bodies could be a static black hole. However, Damour-Schafer[3] exact solutions do not include possible relativistic rotational effects, in particular the spin-orbit interaction. Later on, Esteban and Diaz [4] presented solutions for periastron shift taking the spin-orbit interaction into consideration. However, Esteban and Diaz [4] treated the problem in the test-particle approximation. Our aim, in this letter, is to present a short exposure on the periastron shift when the components in a binary can be of comparable mass and one component, namely, the central body, is spinning. We shall employ the effective one-body method developed and applied to the case of a static central body by Fiziev and Todorov [5] and later extended and applied by Faruque [6] to the case of a spinning central body.

The main ingredient of the effective one-body approach is to use an energy-dependent reduced mass for the orbiter and to consider the gravitational field as generated by both the bodies. As such it is a deformed Kerr field (or a Schwarzchild field) with a coupling parameter $\alpha_G = (Gm_1m_2/c)$ in place of Gm/c^2 in the case of a field generated by the central body alone. For a detailed discussion we refer the reader to [5-6].

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When the central body is a slowly spinning massive body, the equation of motion for the effective particle is given by [6]

$$-\frac{dy}{d\varphi} = [\alpha y^3 - \gamma y^2 + \rho y - \beta]^{1/2} \tag{1}$$

where

$$y = \frac{J}{r}, \quad \rho = \frac{2\alpha_G}{J}, \quad \mu = \frac{a}{J}, \quad \beta = 1- \in^2$$
$$\alpha = (1 - 2\mu \in +3\mu^2 \in^2), \quad \gamma = (1 + 3\beta\mu^2) \tag{2}$$

Here, J is the component of orbital angular momentum of the effective particle along the axis of symmetry of the Kerr field, $a$is the specific spin of the Kerr source, $\in$, the dimensionless energy of the effective particle, is given by

$$\in = \frac{E_w}{m_w} = \frac{w^2 - m_1^2 - m_2^2}{2m_1 m_2}, \tag{3}$$

where E_w is the effective particle c.m. energy and m_w is the energy-dependent reduced mass of the effective particle given by

$$m_w = \frac{m_1 m_2}{w}, \tag{4}$$

and w being the total c.m. energy of the two-body system. Moreover, $r = Rm_w$, R being the invariant distance between the bodies in c.m. frame, and ϕ is the azimuth angle. The orbits under consideration are on the equatorial plane. (Note that we are using geometrical units with $G = c$=1.)

The shape of the orbits defined by Eq.(1) is in general elliptical with the two roots y_1, y_2 of the polynomial

$$P_3(y) = \alpha y^3 - \gamma y^2 + \rho y - \beta = \alpha(y - y_0)(y - y_1)(y - y_2) \tag{5}$$

giving the periastron and apoastron. Note that $y_0$is the largest root, and the three roots satisfy

$$y_0 + y_1 + y_2 = \frac{\gamma}{\alpha}$$
$$\alpha y_0 y_1 y_2 = \beta \tag{6}$$

Motion of the effective particle takes place in the range $y_2 \leq y \leq y_1$ for which $P_3(y)$ is non-negative. As shown in [6], circular orbits are characterized by

$$y_1 = y_2 = y_c = \frac{\gamma(1 - (1 - \frac{3\alpha\rho}{\gamma^2})^{1/2})}{3\alpha} \tag{7}$$

In an elliptic orbit, the periastron shift can be found once the roots y_0, y_1, y_2 in the general case are identified. This task is extremely difficult because of the appearance of four parameters

α,β,ρ and γ in the polynomial. In the Schwarzchild case discussed by Fiziev and Todorov [5], the task is easier because there appears only two parameters, namely, ρ and β. Nevertheless, we attempt to find the dominant term in the periastron shift with spin-orbit interaction present. We can do this easily in a particular case when the orbit is a slightly deformed circle with a very small eccentricity. In that situation we can assume that the roots y_1, y_2 are given by

$$y_1 = y_c + \lambda \tag{8}$$

$$y_2 = y_c - \lambda \tag{9}$$

where $\lambda << 1$ and it is connected to the eccentricity of the orbit. It follows then that

$$y_0 = \frac{\gamma}{\alpha} - (y_1 + y_2) = \frac{\gamma}{\alpha} - 2y_c = \frac{\gamma}{\alpha} - \frac{2\gamma}{3\alpha}\{1 - (1 - \frac{3\rho\alpha}{\gamma^2})^{1/2}\} \tag{10}$$

For elliptic orbits, $(3\rho\alpha/\gamma^2) < 1$, (see Ref.[6]) so we can expand the radical in Eq.(10) using the binomial theorem and get the following well approximated value:

$$y_0 \cong \frac{\gamma}{\alpha} - \frac{\rho}{\gamma} - \frac{3}{4}\frac{\alpha\rho^2}{\gamma^3} \quad . \tag{11}$$

The solution of Eq.(1) can be expressed in terms of elliptic sine functions (see Ref.[5]): For $y(0) = y_1$, we have

$$\frac{y(\varphi) - y_2}{y_1 - y_2} = sn^2(K - \sqrt{\alpha(y_0 - y_2)}\frac{\varphi}{2}, k). \tag{12}$$

The module square k^2 of the elliptic functions is expressed as a ratio of the differences of roots of P_3:

$$k^2 = \frac{y_1 - y_2}{y_0 - y_2} = \frac{2\lambda}{\frac{\gamma}{\alpha} - \frac{3\rho}{2\gamma} - \frac{9}{8}\frac{\alpha\rho^2}{\gamma^3} + \lambda} \tag{13}$$

which, provided $\lambda\alpha < 1$, which is indeed so, can be reduced to

$$k^2 \cong \frac{2\alpha\lambda}{\gamma}(1 + \frac{1}{2}\frac{3\alpha\rho}{\gamma^2}). \tag{14}$$

Now, 4K(k^2) is the real period of *sn(x ,k)* :

$$K = \int_0^1 \frac{dx}{(1 - x^2)(1 - k^2x^2)} = \frac{\pi}{2}F(\frac{1}{2}, \frac{1}{2}; 1; k^2). \tag{15}$$

Next, the change $\Delta\varphi$ of ϕ for a full turn of the effective particle in its orbit and the shift $\delta\varphi$ are given by [5]

$$\Delta\varphi = 2\int_{y_2}^{y_1} \frac{dy}{\sqrt{P_3(y)}} = \frac{4K}{\sqrt{\alpha(y_0 - y_2)}} = 2\pi + \delta\varphi. \tag{16}$$

We can approximate $4K$ by

$$\begin{aligned}4K &= 2\pi(1+\frac{k^2}{4}+\frac{9}{64}k^4+O(k^6)) \\ &\cong 2\pi[1+\frac{1}{2}\frac{\alpha\lambda}{\gamma}(1+\frac{1}{2}\frac{3\alpha\rho}{\gamma^2})+\frac{9}{16}\frac{\alpha^2\lambda^2}{\gamma^2}] \\ &\approx 2\pi(1+\frac{1}{2}\frac{\alpha\lambda}{\gamma}). \end{aligned} \tag{17}$$

where we have neglected terms $\propto \alpha^2\rho\lambda$ and terms $\propto \lambda^2$ since these are really very small. Now, we can approximate $[\alpha(y_0-y_2)]^{-\frac{1}{2}}$ by

$$[\alpha(y_0-y_2)]^{-\frac{1}{2}} \cong \frac{1}{\sqrt{\gamma}}(1+\frac{1}{4}\frac{3\alpha\rho}{\gamma^2}-\frac{1}{2}\frac{\alpha\lambda}{\gamma}). \tag{18}$$

Finally, using Eqs. (17) and (18) in Eq.(16), we obtain

$$\delta\varphi \cong 2\pi(\frac{1}{\sqrt{\gamma}}-1)+\frac{3\pi}{2}\frac{\alpha\rho}{\sqrt{\gamma}\gamma^2}. \tag{19}$$

which is the dominant contribution to periastron shift of an effective particle in Kerr field in case the orbit is an elliptic one with very small eccentricity such that we can approximate the periastron and apoastron distances from the focus by Eqs.(8) and (9) and if we use the effective one-body approach to relativistic two-body problem developed by Fiziev and Todorov [5] and later extended and applied to the Kerr metric case by Faruque[6]. Now, we are in a position to make further simplifications in the parameters μ , γ and ρ. For specific spin of the central body much smaller than the orbital angular momentum of the orbiter, we have that $\mu = (a/J) << 1$ and we can approximate $\gamma \approx 1, \alpha \approx \rho(1-2\mu \in)$ and we obtain

$$\delta\varphi \cong \frac{3\pi}{2}\rho^2(1-2\mu \in). \tag{20}$$

For $a = 0$, i.e., for the Schwarzchild case, we obtain

$$\delta\varphi(S) \cong \frac{3\pi}{2}\rho^2 \tag{21}$$

which is the dominant term in periastron shift in the Schwarzchild field found by Fiziev and Todorov [5]. Therefore, our results, Eqs.(19) –(21) are in good parallel with the results of the case of non-spinning central body. We now wish to point out that there are in fact two cases corresponding to parallel and anti parallel spin and orbital angular momentum of the bodies in the binary. To distinguish these two possibilities, we note that $\mu = a/J = -\left|{}^{a}\!/_{J}\right|$ for the anti parallel case and $\mu = \left|{}^{a}\!/_{J}\right|$ for the parallel case. There fore, Eq.(20) can be separated as

$$\delta\varphi \cong \frac{3\pi}{2}\rho^2(1+2\left|\frac{a}{J}\right| \in), \qquad \text{anti-parallel spin and orbital angular momentum} \tag{22}$$

$$\delta\varphi \cong \frac{3\pi}{2}\rho^2(1-2\left|\frac{a}{J}\right| \in), \qquad \text{parallel spin and orbital angular momentum} \tag{23}$$

In words, the periastron shift is more in case the spin and orbital angular momentum respectively of the central body and the orbiter are anti-parallel than it is in case the later are parallel. That is, the spin-orbit interaction diminish (increases) the value of the periastron shift when, for example, a star in a binary system orbits in the same (opposite) direction as the central body which can be a rotating black hole. This property of the periastron shift in Kerr field is in full agreement with what is obtained by Esteban and Diaz [4] using the test-particle approximation.

In conclusion, we have calculated the periastron shift of an effective particle orbiting a slowly spinning central body using the effective one-body approach to the relativistic two-body problem of Fiziev and Todorov [5] and of Faruque [6], and have found that the dominant contribution to periastron shift depends on the magnitude and direction of the spin of the central body and orbital angular momentum of the orbiter. Our result agrees well with the dominant term for the periastron shift in case of static central body (that is, with the Schwarzchild case). Moreover, the qualitative property of the periastron shift in Kerr field which we have found in this work is in accord with that found earlier by other authors by different procedure [4].

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