PROBLEMS WITH THE GAUSSIAN STATISTICS IN STOCHASTIC THEORIES OF THE RADIATIVE TRANSFER

M. M. Selim, V. Bezák[1]
Department of Solid State Physics, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava 84248, Slovakia

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We present a stochastic non-perturbative analysis of the 'rod model' of the theory of radiative transfer treating the 'rods' of lengths L as random media. The cross-section function $\sigma(x)$ (per unit length) is defined as a Gaussian random function with the mean value $\bar{\sigma}$, correlation length ℓ and variance η_σ^2. To characterize the random media, we introduce the parameter $\beta = \eta_\sigma \sqrt{\ell/\bar{\sigma}}$. If L is fixed, the optical length $X = \int_0^L \mathrm{d}x\, \sigma(x)$ becomes random. Then also the values of the transmission and reflection coefficient, $\mathcal{T}(X)$ and $\mathcal{R}(X)$, are random. Since $X > 0$ for physical reasons, we define the probability density of X as a Gaussian only for $X > 0$; if $X < 0$, we define it as zero. (We show that if such a cut off were not considered and if the Gaussian distribution were extrapolated on the negative semiaxis of X, the stochastic analysis of the albedo problem would bear non-physical divergencies.) With this probability density, we carry out (both analytical and numerical) calculations of the first-order and second-order statistical moments of $\mathcal{T}$ and $\mathcal{R}$, as well as of the variances of $\mathcal{T}$ and $\mathcal{R}$, as functions of the averaged optical length $\bar{\sigma}L$ and of the parameter β. We calculate also the correlation function between the stochastic values of $\mathcal{T}$ and the stochastic values of $\mathcal{R}$.

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1 Introduction

During the last century, great strides have been achieved in solving various kinetic equations of the radiative transfer in media defined as deterministic objects [1 - 3]. But then, if attention is focused to *random* media, the radiative transfer theory has to cope with new problems. When the transport of specified particles − such as the neutrons, say − is considered in random media (cf. e.g. [4 - 9]), the densities of the microscopic targets on which the particles scatter (or in which they may become absorbed causing eventually emission of daughter particles), have to be taken as random functions.

In the present paper, we confine ourselves to a simplified version of the radiative transfer problem, assuming that the neutrons impact upon nuclei of a given kind. We will formulate the radiative transfer as a one-speed problem. For the sake of simplicity, we will adopt a one-dimensional model defining the density of the targets as a random function of one variable, x. The product of this density with the differential cross-section for the impacts of the particles on these targets defines a random function $\sigma(x) \geq 0$. Each impact may cause, with a probability

[1]E-mail address: bezak@fmph.uniba.sk

$c > 0$, emission of a daughter neutron which may be shot either forwards or backwards with the same probability (equal to $\frac{1}{2}$). (If we were treating the radiative transfer of other particles than the neutrons, we could give the parameter c another interpretation.) What we have just described has been called the 'rod model' in the theory of the radiative transfer (cf. e.g. [10 - 13]). The rod model is instructive but oversimplifying intentionally the problem.

Adopting the rod model, we pay heed to the albedo theory. This implies that the medium under consideration (the rod) is defined in a finite interval. We define $\sigma(x)$ for $0 < x < L$, assuming that the intervals $x < 0$ and $x > L$ correspond to vacuum. A beam of the particles with a given velocity v enters the rod from the left. Two beams are emitted from the rod: one to the right ($x > L$) with a probability $\mathcal{T}$, and one to the left ($x < 0$) with a probability $\mathcal{R}$. Since $\sigma(x)$ is defined as a random function, the quantities $\mathcal{T}$ and $\mathcal{R}$ are random.

In [14], one of us has calculated, with $\sigma(x)$ defined as a *Gaussian* random function, the statistical moments $\langle \mathcal{T}^n \rangle$ and $\langle \mathcal{R}^n \rangle$ for n = 1, 2 in a way typical for a perturbation theory. In the present paper, we will follow a non-perturbative way. Again, we will confine ourselves to discuss $\sigma(x)$ as a Gaussian random function. In fact, the Gaussianity of $\sigma(x)$ is a very natural assumption which was also accepted by other authors in various theories of the radiative transfer (cf. e.g. [6]). Nevertheless, we will show in the present paper that the Gaussianity implies some unexpected consequencies for the theory of the radiative transfer. Their origin is clarified in Sections 2 and 3. To exemplify the problem, we calculate (in Section 3) the average values of $\langle \mathcal{T} \rangle$ and $\langle \mathcal{R} \rangle$ and (in Section 4) the average values of $\langle \mathcal{T}^2 \rangle$ and $\langle \mathcal{R}^2 \rangle$. In Section 5, we present some results of our calculations graphically.

2 Gaussian distribution function with a one-side cut off

In thermodynamics, when dealing with fluctuations, one uses almost exclusively Gaussian random variables [15]. According to the theory of probabilities, the Gaussianity of thermodynamic fluctuations is well supported by the validity of the Central Limit Theorem. On the other hand, the postulate of the Gaussianity allows us to perform some calculations in a much easier way than with non-Gaussian random variables.

Let X be a real random variable and $\bar{X} = \langle X \rangle > 0$. (We use the angular brackets to denote the averaging.) To define X as a Gaussian random variable, we have to assume that values of X may span the whole real axis and we have to require the existence of the variance $\eta^2 = \langle (X - \bar{X})^2 \rangle > 0$. With the Gaussian probability density

$$P_{\bar{X}}^{\mathrm{G}}(X, \eta^2) = \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(- \frac{(X - \bar{X})^2}{2\eta^2} \right), \tag{1}$$

it is easy to verify that

$$\begin{aligned} Z^{\mathrm{G}}(\kappa, \eta^2) &= \langle \exp[\kappa(X - \bar{X})] \rangle \\ &= \int_{-\infty}^{\infty} \mathrm{d}X \, \exp[\kappa(X - \bar{X})] \, P_{\bar{X}}^{\mathrm{G}}(X, \eta^2) = \exp\left(\frac{\kappa^2 \eta^2}{2} \right), \end{aligned} \tag{2}$$

where κ may be an arbitrary complex number. If $\kappa = \mathrm{i}k$ with real k, the function $Z(\mathrm{i}k, \eta^2)$ represents what probabilists call the characteristic function of the Gaussian distribution (cf. e.g.

[12]). (Whichever the characteristic function $Z(\mathrm{i}k)$ is, it determines the corresponding probability density P.)

Often it may happen that X is a physical quantity for which we have to respect a constraint. Throughout the present paper, we assume that $X > 0$. If $0 < \eta \ll \bar{X}$, the probabilistic distribution of X, excluding the negative values of X, may still be very close to the Gaussian. Indeed, employing the unit step function $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$, we may define the probability density

$$P^{\geq}_{\bar{X}}(X, \eta^2) = \frac{\Theta(X)\, P^{\mathrm{G}}_{\bar{X}}(X, \eta^2)}{\int_0^\infty \mathrm{d}X'\, P^{\mathrm{G}}_{\bar{X}}(X', \eta^2)} \,. \tag{3}$$

Clearly,

$$\int_0^\infty \mathrm{d}X'\, P^{\mathrm{G}}_{\bar{X}}(X', \eta^2) = \left[\int_{-\infty}^{\infty} - \int_{-\infty}^{-\bar{X}}\right] \mathrm{d}X'\, P^{\mathrm{G}}_0(X', \eta^2)$$
$$= 1 - \int_{\bar{X}}^\infty \mathrm{d}X\, P^{\mathrm{G}}_0(X, \eta^2) = 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\bar{X}}{\sqrt{2\eta^2}}\right) .$$

Here we have used the standard denotation: $\mathrm{erfc}(u) = (2/\sqrt{\pi}) \int_u^\infty \mathrm{d}t \exp(-\,t^2)$. When regarding the asymptotic expression $\mathrm{erfc}(u) \approx \exp(-\,u^2)/(\sqrt{\pi}u)$ [17], we observe that the difference of the denominator in expression (3) from unity is negligible if $0 < \eta \ll \bar{X}$. Hence, if we have to carry out the averaging of a function $F(X)$ defined only for $X > 0$, we may use the integral

$$\langle F(X) \rangle = \int_0^\infty \mathrm{d}X\, F(X)\, P^{\geq}_{\bar{X}}(X, \eta^2) \,.$$

If the function $F(X)$ does not lose its formal mathematical regularity when negative (non-physical) values of X are admitted in it, we may legitimately use the Gaussian distribution function, since the perpetrated error is negligible:

$$\langle F(X) \rangle \approx \int_0^\infty \mathrm{d}X\, F(X)\, P^{\mathrm{G}}_{\bar{X}}(X, \eta^2) \,.$$

Surprisingly enough, the situation becomes utterly different in theories of the radiative transfer. As we will show in Section 3 discussing the rod model, we have to average functions $F(X)$ which, being defined for $X > 0$, do not behave regularly if X is allowed to run (formally) along the negative semiaxis. Then the use of the Gaussian probability density $P^{\mathrm{G}}_{\bar{X}}(X, \eta^2)$ for X manifests its Achilles's heel since the integral

$$\int_{-\infty}^\infty \mathrm{d}X\, F(X)\, P^{\mathrm{G}}_{\bar{X}}(X, \eta^2)$$

diverges.

Apparently, the most straightforward corrective is to use the probability density $P^{\geq}_{\bar{X}}(X, \eta^2)$ instead of $P^{\mathrm{G}}_{\bar{X}}(X, \eta^2)$:

$$\langle F(X) \rangle = \int_{-\infty}^\infty \mathrm{d}X\, F(X)\, P^{\geq}_{\bar{X}}(X, \eta^2) \approx \int_0^\infty \mathrm{d}X\;\, F(X)\, P^{\mathrm{G}}_{\bar{X}}(X, \eta^2) \,. \tag{4}$$

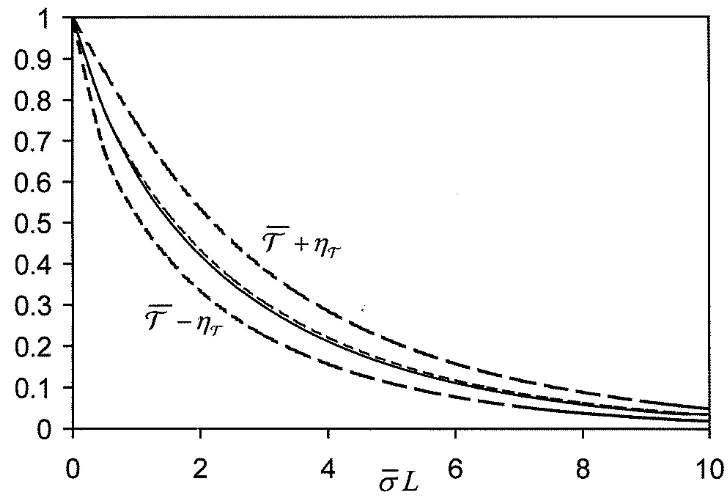

Fig. 1. The most probable values of the transmission coefficient $\mathcal{T}$ for given values of the averaged optical length $\bar{\sigma}L$ and for $c = 0.9$. The full line corresponds to $\beta = 0$. The dash lines correspond to $\beta = 0.3$. The random values of $\mathcal{T}$ are distributed around $\bar{\mathcal{T}}$ and lie mainly (with a probability approximately equal to 0.7) in the interval $(\bar{\mathcal{T}} - \eta_{\mathcal{T}}, \bar{\mathcal{T}} + \eta_{\mathcal{T}})$. The values of $\bar{\mathcal{T}}$ for $\beta = 0.3$ are very near to the values of $\mathcal{T}$ for $\beta = 0$.

When neglecting the difference between $1 - \frac{1}{2}\mathrm{erf}\,(\bar{X}/\sqrt{2\eta^2})$ and unity, we may reinterpret formula (4) as the *Gaussian averaging* of the function $\Theta(X)F(X)$:

$$\langle \Theta(X)F(X)\rangle = \int_0^\infty \mathrm{d}X\, F(X)\, P_{\bar{X}}^{\mathrm{G}}(X,\eta^2)\,. \tag{5}$$

The averaged value of $\exp[\kappa(X-\bar{X})]$ with the probability density $P_{\bar{X}}^{\gtrless}(X,\eta^2)$ is given by the integral

$$Z_{\bar{X}}^{\gtrless}(\kappa,\eta^2) = \int_0^\infty \mathrm{d}X\, \exp[\kappa(X-\bar{X})]\, P_{\bar{X}}^{\gtrless}(X,\eta^2)$$

$$= \frac{1}{1-\frac{1}{2}\,\mathrm{erfc}\,\left(\bar{X}/\sqrt{2\eta^2}\right)}\, \frac{1}{\sqrt{2\pi\eta^2}} \int_0^\infty \mathrm{d}X\, \exp[\kappa(X-\bar{X})]\, \exp\left(-\,\frac{(X-\bar{X})^2}{2\eta^2}\right)$$

$$= \frac{\exp(\kappa^2\eta^2/2)}{1-\frac{1}{2}\,\mathrm{erfc}\,\left(\bar{X}/\sqrt{2\eta^2}\right)} \left[1 - \frac{1}{\sqrt{2\pi\eta^2}} \int_{\bar{X}+\kappa\eta^2}^\infty \mathrm{d}X'\, \exp\left(-\,\frac{X'^2}{2\eta^2}\right)\right].$$

Thus,

$$Z_{\bar{X}}^{\gtrless}(\kappa,\eta^2) = \frac{1-\frac{1}{2}\,\mathrm{erfc}\,\left[\left(\bar{X}+\kappa\eta^2\right)/\sqrt{2\eta^2}\right]}{1-\frac{1}{2}\,\mathrm{erfc}\,\left(\bar{X}/\sqrt{2\eta^2}\right)}\, \exp\left(\frac{\kappa^2\eta^2}{2}\right).$$

If $0 < \eta \ll \bar{X}$, we may approximate the denominator as unity. Since $\mathrm{erfc}(u) = 1 - \mathrm{erf}(u)$, we may finally write

$$Z_{\bar{X}}^{\gtrless}(\kappa,\eta^2) \approx \frac{1+\mathrm{erf}\,\left[\left(\bar{X}+\kappa\eta^2\right)/\sqrt{2\eta^2}\right]}{2}\, \exp\left(\frac{\kappa^2\eta^2}{2}\right). \tag{6}$$

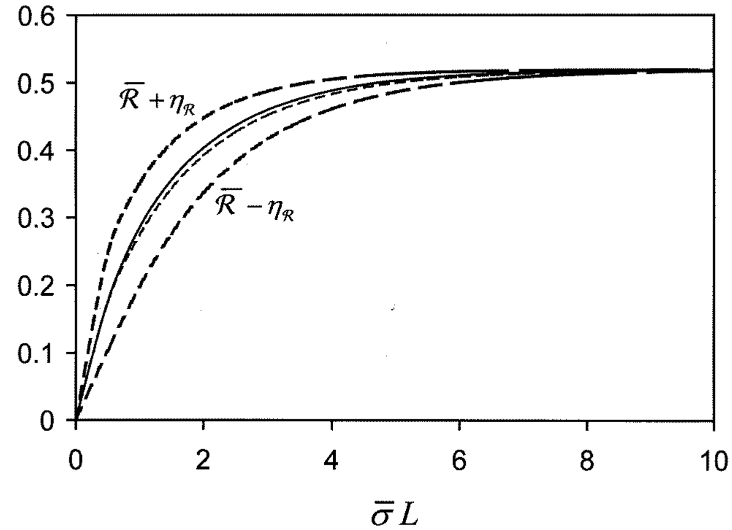

Fig. 2. The most probable values of the reflection coefficient $\mathcal{R}$ for given values of the averaged optical length $\bar{\sigma}L$ and for $c = 0.9$. The full line corresponds to $\beta = 0$. The dash lines correspond to $\beta = 0.3$. The random values of $\mathcal{R}$ are distributed around $\bar{\mathcal{R}}$ and lie mainly (with a probability approximately equal to 0.7) in the interval $(\bar{\mathcal{R}} - \eta_{\mathcal{R}}, \bar{\mathcal{R}} + \eta_{\mathcal{R}})$. The values of $\bar{\mathcal{R}}$ for $\beta = 0.3$ are very near to the values of $\mathcal{R}$ for $\beta = 0$.

As the error function is odd, $\mathrm{erf}(u) = (2/\sqrt{\pi}) \int_0^u \mathrm{d}t \exp(-t^2) = -\mathrm{erf}(-u)$, the asymptotic behaviour of the function $Z_{\bar{X}}^{\gtrless}(\kappa, \eta^2)$ for $\kappa \to \infty$ differs from the behaviour for $\kappa \to -\infty$:

$$Z_{\bar{X}}^{\gtrless}(\kappa, \eta^2) \to \infty \quad \text{if} \quad \kappa \to \infty \tag{7a}$$

and

$$Z_{\bar{X}}^{\gtrless}(\kappa, \eta^2) \to 0 \quad \text{if} \quad \kappa \to -\infty\,. \tag{7b}$$

This behaviour contrasts radically with the behaviour of the function $Z^{\mathrm{G}}(\kappa, \eta^2)$ defined by formula (2):

$$Z^{\mathrm{G}}(\kappa, \eta^2) \to \infty \quad \text{if} \quad |\kappa| \to \infty\,.$$

3 The probabilities $\mathcal{T}$ and $\mathcal{R}$ of the rod model

The stationary kinetics of the rod model is determined by the equations

$$\frac{\mathrm{d}\phi^+(x)}{\mathrm{d}x} + \sigma(x)\,\phi^+(x) = \frac{c}{2}\,\sigma(x)\,\left[\phi^+(x) + \phi^-(x)\right], \tag{8^+}$$

$$-\frac{\mathrm{d}\phi^-(x)}{\mathrm{d}x} + \sigma(x)\,\phi^-(x) = \frac{c}{2}\,\sigma(x)\,\left[\phi^+(x) + \phi^-(x)\right]. \tag{8^-}$$

In this model, the velocities of all particles are the same in their absolute value. The functions $\phi^+(x)$ and $\phi^-(x)$ mean the probability densities of these particles, the first for the particles travelling with the positive velocity and the second for the particles travelling with the negative velocity. We assume that c is a constant, $0 < c < 1$. The position variable x runs from zero to $L > 0$. In the albedo problem, $\phi^+(0)$ is a given (positive) constant and $\phi^-(L) = 0$. The values of $\phi^+(L)$ and $\phi^-(0)$ represent, respectively, the intensities of the forward and backward radiation of the rod. Correspondingly, the coefficients of the forward and backward radiation are defined as the ratios:

$$\mathcal{T} = \frac{\phi^+(L)}{\phi_0^+}\,, \qquad \mathcal{R} = \frac{\phi^-(0)}{\phi_0^+}\,.$$

Both these coefficients are simple functions of the 'optical length' X of the rod,

$$X = \int_0^L \mathrm{d}x\, \sigma(x)\,. \tag{9}$$

According to detailed calculations presented in [14],

$$\mathcal{T} \;\equiv\; \mathcal{T}(X) = (1 - \alpha^2)\, \frac{\exp(-\gamma X)}{1 - \alpha^2 \exp(-2\gamma X)} \tag{10}$$

and

$$\mathcal{R} \;\equiv\; \mathcal{R}(X) = \alpha\, \frac{1 - \exp(-2\gamma X)}{1 - \alpha^2 \exp(-2\gamma X)}\,, \tag{11}$$

$$\gamma = \sqrt{1-c}\,, \tag{12}$$

$$\alpha = \frac{1-\gamma}{1+\gamma} = \frac{1 - \sqrt{1-c}}{1 + \sqrt{1-c}}\,. \tag{13}$$

If the function $\sigma(x)$ is such that the optical length X is positive, then formulae (10) and (11) are valid quite generally. And if $\sigma(x)$ is a random function, X becomes a random variable and, consequently, the values of $\mathcal{T}(X)$ and $\mathcal{R}(X)$ are also random.

We define $\sigma(x)$ as a Gaussian random function with the constant mean value $\bar{\sigma}$ and the constant variance η_σ^2:

$$\bar{\sigma} = \langle \sigma(x) \rangle\,, \qquad \eta_\sigma^2 = \langle [\sigma(x) - \bar{\sigma}]^2 \rangle\,. \tag{14}$$

The latter value is the 'diagonal element' of the autocorrelation function:

$$W_\sigma(|x_1 - x_2|) = \langle [\sigma(x_1) - \bar{\sigma}][\sigma(x_2) - \bar{\sigma}] \rangle\,, \qquad W(0) = \eta_\sigma^2\,. \tag{15}$$

Various Gaussian random functions may have various autocorrelation functions. As the form of the function $W(|x|)$ is not important for our further derivations, we will not specify it.

The linearity of relation (9) implies that if $\sigma(x)$ is Gaussian, then X is also Gaussian (cf. e.g. [16]). The mean value of X is equal to

$$\bar{X} = \langle X \rangle = \int_0^L \mathrm{d}x\, \langle \sigma(x) \rangle = \bar{\sigma} L\,. \tag{16}$$

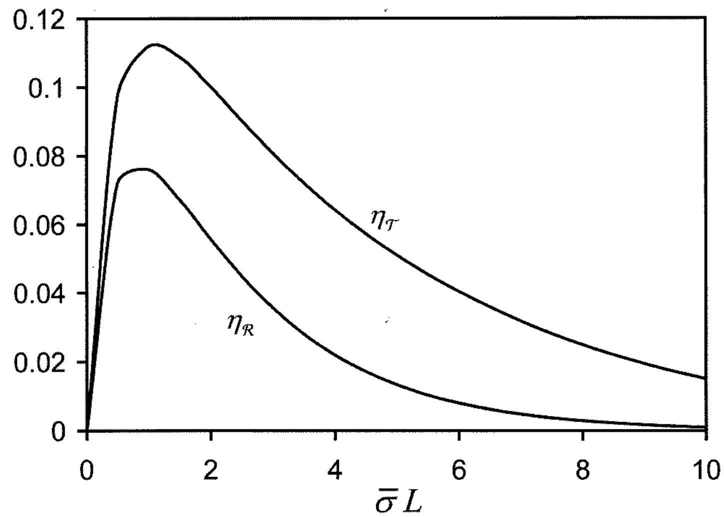

Fig. 3. The dependence of the r.m.s. deviation $\eta_{\mathcal{T}}$ of $\mathcal{T}$ (the upper curve) and the r.m.s. deviation $\eta_{\mathcal{R}}$ of $\mathcal{R}$ (the lower curve) on $\bar{\sigma}L$. Both the curves correspond to $c = 0.9$ and $\beta = 0.3$.

The variance η^2 of X depends on the form of the function $W(|x|)$:

$$\eta^2 = \langle (X - \bar{X})^2 \rangle = \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2 \, \langle [\sigma(x_1) - \bar{\sigma}][\sigma(x_2) - \bar{\sigma}] \rangle$$

$$= \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2 \, W(|x_1 - x_2|) \, . \tag{17}$$

We refrain from calculating the parameter η and take simply both $\bar{X} > 0$ and $\eta > 0$ as given constants. We require that

$$0 < \eta \ll \bar{X} \, . \tag{18}$$

With this assumption, we approximate X as a Gaussian random variable, respecting the zero value of the probability density of X for $X < 0$, as it has been elucidated in Section 2.

In other words, we define the averaging not with the probability density $P^{\mathrm{G}}_{\bar{X}}(X)$ but with the probability density $P^{\geq}_{\bar{X}}(X)$. The proper reason for such a pedantry descends from the forms of the functions $\mathcal{T}(X)$ and $\mathcal{R}(X)$ since these functions exhibit one single pole at a point $X_{\mathrm{p}} < 0$ that is determined by the equation

$$1 - \alpha^2 \exp(-2\gamma X_{\mathrm{p}}) = 0 \, .$$

This equation yields the value

$$X_{\mathrm{p}} = \frac{\ln \alpha}{\gamma} = \frac{1}{\gamma} \ln \frac{1 - \gamma}{1 + \gamma} = \frac{1}{\sqrt{1 - c}} \ln \frac{1 - \sqrt{1 - c}}{1 + \sqrt{1 - c}} < 0 \, . \tag{19}$$

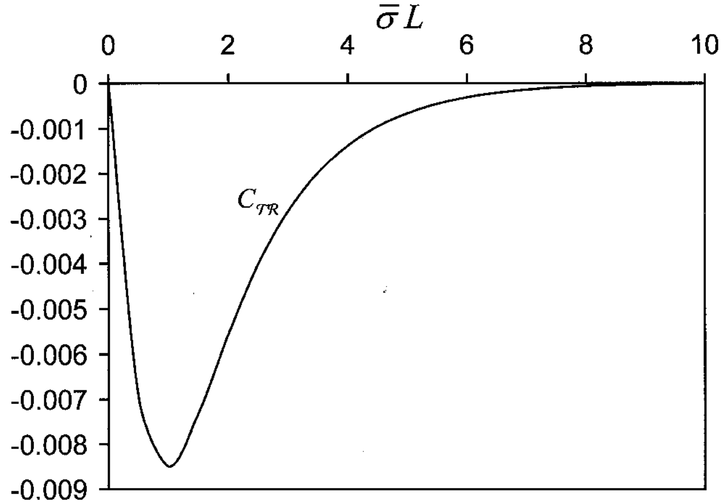

Fig. 4. The dependence of the off-diagonal element $\langle(\mathcal{T}-\bar{\mathcal{T}})(\mathcal{R}-\bar{\mathcal{R}})\rangle$ of the correlation matrix $\mathbf{C}$ on $\bar{\sigma}L$ for $c = 0.9$ and $\beta = 0.3$.

Note that

$$0 < \alpha < 1\,, \tag{20}$$

for $0 < c < 1$ and $0 < \gamma < 1$.

If $X > X_{\rm p}$, then

$$\frac{1}{1\,-\,\alpha^2\exp(-\,2\gamma X)} = \sum_{n=0}^{\infty}\,\alpha^{2n}\,\exp(-\,2n\gamma X)\,. \tag{21}$$

On the other hand, if $X < X_{\rm p}$, the geometrical series (21) diverges. This is just the reason for which the Gaussian probability density function $P^{\rm G}_{\bar{X}}(X,\eta^2)$ fails when statistical moments of the functions $\mathcal{T}(X)$ and $\mathcal{R}(X)$ (including the average values $\langle\mathcal{T}(X)\rangle$ and $\langle\mathcal{R}(X)\rangle$) are calculated.

If one employs the Gaussian probability density with the cut off, $P^{\gtrless}_{\bar{X}}(X,\eta^2)$, there is no problem with the divergency of series (21).

So, according to formula (10), we write

$$\mathcal{T}(X) = (1-\,\alpha^2)\,\sum_{n=0}^{\infty}\,\alpha^{2n}\,\exp[-\,(2n+1)\gamma X]$$

and

$$\bar{\mathcal{T}}\;\equiv\;\langle\mathcal{T}(X)\rangle = (1-\,\alpha^2)\,\sum_{n=0}^{\infty}\,\alpha^{2n}\,\langle\,\exp[-\,(2n+1)\gamma X]\rangle$$

$$= (1-\,\alpha^2)\,\sum_{n=0}^{\infty}\,\alpha^{2n}\,\int_0^{\infty}{\rm d}X\,\exp[-\,(2n+1)\gamma X]\,P^{\gtrless}_{\bar{X}}(X,\eta^2)\,.$$

The integrals in this series can be calculated in the same way as we have calculated the function $Z^{>}(\kappa, \eta^2)$ in Section 2. For the integral in the nth term, when taking $\kappa = -(2n+1)\gamma$, we may use formula (6). Thus we obtain the series

$$\bar{\mathcal{T}} \approx \frac{(1-\alpha^2)}{2} \sum_{n=0}^{\infty} \alpha^{2n} \operatorname{erfc}\left(\frac{(2n+1)\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}}\right) \times \exp\left(-(2n+1)\gamma\bar{X} + \frac{(2n+1)^2\gamma^2\eta^2}{2}\right). \qquad (22)$$

Similarly, after arranging formula (11) in the form

$$\mathcal{R}(X) = \frac{1}{\alpha}\left[1 - \frac{1-\alpha^2}{1-\alpha^2\exp(-2\gamma X)}\right],$$

we can write down the series

$$\mathcal{R}(X) = \frac{1}{\alpha}\left[1 - (1-\alpha^2)\sum_{n=0}^{\infty} \alpha^{2n} \exp(-2n\gamma X)\right]$$

and

$$\bar{\mathcal{R}} \equiv \langle \mathcal{R}(X) \rangle = \frac{1}{\alpha}\left[1 - (1-\alpha^2)\sum_{n=0}^{\infty}\alpha^{2n}\langle \exp(-2n\gamma X)\rangle\right]$$
$$= \frac{1}{\alpha}\left[1 - (1-\alpha^2)\sum_{n=0}^{\infty}\alpha^{2n}\int_0^{\infty} \mathrm{d}X \exp(-2n\gamma X)\, P_{\bar{X}}^{\gtrless}(X,\eta^2)\right].$$

Utilizing formula (6) with $\kappa = -2n\gamma$, we arrive at the result

$$\bar{\mathcal{R}} \approx \frac{1}{\alpha}\left[1 - \frac{1-\alpha^2}{2}\sum_{n=0}^{\infty}\alpha^{2n}\operatorname{erfc}\left(\frac{2n\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}}\right)\exp(-2n\gamma\bar{X} + 2n^2\gamma^2\eta^2)\right]. \qquad (23)$$

With the fulfilment of inequalities (18), the convergence of the series (22) and (23) may be slow. Let ν be the root of the equation $2\nu\gamma\eta^2 - \bar{X} = 0$. Clearly, $\nu = \bar{X}/(2\gamma\eta^2)$. We estimate the 'critical index' n_{c} as the integer part of ν:

$$n_{\mathrm{c}} = \text{integer part of } \frac{\bar{X}}{2\gamma\eta^2}. \qquad (24)$$

Thus, n_{c} may be a relatively high number. If $n > 2n_{\mathrm{c}}$, say, we may consider the asymptotic expression for the complementary error function:

$$\operatorname{erfc}\left(\frac{2n\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}}\right) \approx \frac{1}{\sqrt{\pi}}\frac{\sqrt{2\eta^2}}{2n\gamma\eta^2 - \bar{X}} \exp\left[-\left(\frac{2n\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}}\right)^2\right].$$

Clearly, the contribution of the terms with $n > 2n_{\mathrm{c}}$ is negligible, being proportional to the small factor $\exp[-\bar{X}^2/(2\eta^2)]$.

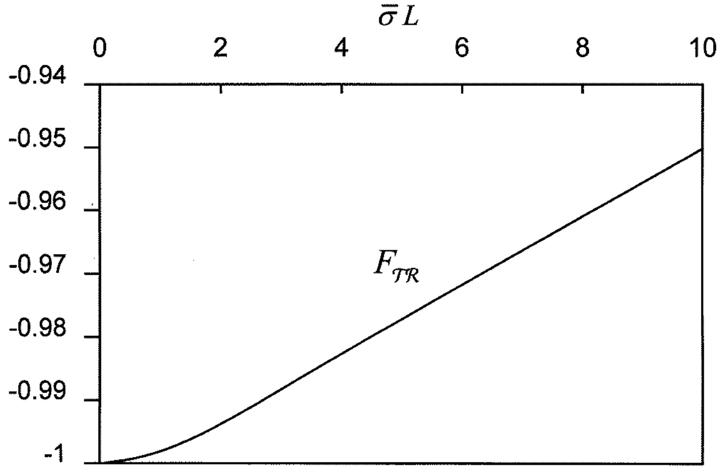

Fig. 5. The correlation function $F_{\mathcal{TR}}(\bar{\sigma}L)$ for $c = 0.9$ and $\beta = 0.3$.

4 The averaging of $\mathcal{T}^2(X)$, $\mathcal{R}^2(X)$ and $\mathcal{T}(X)\mathcal{R}(X)$

As it has been elucidated in the previous Sections, the randomness of the function $\sigma(x)$ implies that the optical length X is a random variable and the randomness of X implies that the values of $\mathcal{T}(X)$ and $\mathcal{R}(X)$ are random. Although we have taken X as a Gaussian random variable, the variables $\mathcal{T}(X)$ and $\mathcal{R}(X)$ are *not* Gaussian. Besides, the random variables $\mathcal{T}(X)$ and $\mathcal{R}(X)$ are correlated. Because of the non-Gaussianity, one can completely comprehend the statistics of $\mathcal{T}(X)$ and $\mathcal{R}(X)$ after calculating *all* statistical moments $\langle \mathcal{T}^i(X)\mathcal{R}^j(X)\rangle$ with i, j =0, 1, 2, ... Here we will confine ourselves to the calculation of the second-order moments $\langle \mathcal{T}^2(X)\rangle$, $\langle \mathcal{R}^2(X)\rangle$ and $\langle \mathcal{T}(X)\mathcal{R}(X)\rangle$. Their values determine the second-order correlation matrix

$$\mathbf{C} = \begin{pmatrix} C_{\mathcal{TT}} & C_{\mathcal{TR}} \\ C_{\mathcal{RT}} & C_{\mathcal{RR}} \end{pmatrix} \tag{25}$$

with the elements

$$C_{\mathcal{TT}} \equiv \eta_{\mathcal{T}}^2 = \langle [\mathcal{T}(X) - \bar{\mathcal{T}}]^2 \rangle = \langle \mathcal{T}^2(X)\rangle - \bar{\mathcal{T}}^2 , \tag{26}$$

$$C_{\mathcal{RR}} \equiv \eta_{\mathcal{R}}^2 = \langle [\mathcal{R}(X) - \bar{\mathcal{R}}]^2 \rangle = \langle \mathcal{R}^2(X)\rangle - \bar{\mathcal{R}}^2 , \tag{27}$$

and

$$C_{\mathcal{TR}} = C_{\mathcal{RT}} = \langle [\mathcal{T}(X) - \bar{\mathcal{T}}][\mathcal{R}(X) - \bar{\mathcal{R}}] \rangle = \langle \mathcal{T}(X)\mathcal{R}(X)\rangle - \bar{\mathcal{T}}\bar{\mathcal{R}} . \tag{28}$$

To calculate the average value of $\mathcal{T}^2(X)$, we start from the series

$$\mathcal{T}^2(X) = (1 - \alpha^2)^2 \left[\sum_{n=0}^{\infty} \alpha^{2n} \exp[-(2n+1)\gamma X] \right]^2$$

$$= (1- \alpha^2)^2 \exp(-2\gamma X) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \alpha^{2(n_1+n_2)} \exp[-2(n_1+n_2)\gamma X]$$

$$= (1- \alpha^2)^2 \exp(-2\gamma X) \sum_{n=0}^{\infty} (n+1)\, \alpha^{2n} \exp(-2n\gamma X)\,.$$

The reduction of the double sum to the single one has followed from the identity

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \Phi_{n_1+n_2} = \sum_{n=0}^{\infty} (n+1)\, \Phi_n\,,$$

which is valid for each function Φ_n of the integer variable n. Within the framework of the approximation tolerating the identification of expression (4) with expression (5), we write

$$\langle \mathcal{T}^2(X) \rangle = (1- \alpha^2)^2 \sum_{n=0}^{\infty} (n+1)\, \alpha^{2n} \left\langle \exp[-2(n+1)\gamma X] \right\rangle$$

$$\approx (1- \alpha^2)^2 \sum_{n=0}^{\infty} (n+1)\, \alpha^{2n} \int_0^{\infty} \mathrm{d}X \, \exp[-2(n+1)\gamma X]\, P_{\bar{X}}^{\mathrm{G}}(X, \eta^2)\,.$$

Since

$$\int_0^{\infty} \mathrm{d}X \, \exp[-2(n+1)\gamma X]\, P_{\bar{X}}^{\mathrm{G}}(X, \eta^2) =$$

$$\frac{1}{2}\, \mathrm{erfc}\left(\frac{2(n+1)\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}} \right) \exp[-2(n+1)\gamma \bar{X} + 2(n+1)^2\gamma^2\eta^2]$$

(cf. Eq. (6)), we have got the expression

$$\langle \mathcal{T}^2(X) \rangle \approx \frac{(1-\alpha^2)^2}{2} \sum_{n=0}^{\infty} (n+1)\alpha^{2n}$$

$$\times\, \mathrm{erfc}\left(\frac{2(n+1)\gamma\eta^2 - \bar{X}}{\sqrt{2\eta^2}} \right) \exp[-2(n+1)\gamma \bar{X} + 2(n+1)^2\gamma^2\eta^2]\,. \qquad (29)$$

Analogically, in order to calculate $\langle \mathcal{R}^2(X) \rangle$, we start from the expression

$$\mathcal{R}^2(X) = \frac{1}{\alpha^2} \left\{ 1 - 2(1-\alpha^2) \sum_{n=0}^{\infty} \alpha^{2n} \exp(-2n\gamma X) \right.$$

$$\left. + (1-\alpha^2)^2 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \alpha^{2(n_1+n_2)} \exp[-2(n_1+n_2)\gamma X] \right\}.$$

As above, we can reduce the double sum to the single one. Then we obtain the series

$$\mathcal{R}^2(X) = \frac{1}{\alpha^2} \left\{ 1 + (1-\alpha^2) \sum_{n=0}^{\infty} [(n+1)(1-\alpha^2) - 2]\, \alpha^{2n} \exp(-2n\gamma X) \right\}.$$

Approximating $\langle \mathcal{R}^2(X)\rangle = \int_0^\infty \mathrm{d}X \mathcal{R}^2(X) P_{\bar{X}}^{\geq}(X,\eta^2)$ by the Gaussian average value of $\Theta(X)\mathcal{R}^2(X)$, we obtain readily the result

$$\langle \mathcal{R}^2(X)\rangle \approx \frac{1}{\alpha^2}\left[1 + \frac{1-\alpha^2}{2}\sum_{n=0}^{\infty}\left[(n+1)(1-\alpha^2)-2\right]\alpha^{2n}\right.$$
$$\left.\times\, \mathrm{erfc}\left(\frac{2n\gamma\eta^2-\bar{X}}{\sqrt{2\eta^2}}\right)\exp(-\,2n\gamma\bar{X} + 2n^2\gamma^2\eta^2)\right]. \tag{30}$$

It remains to calculate the average value of the product $\mathcal{T}(X)\mathcal{R}(X)$. Now we employ the double series

$$\mathcal{T}(X)\mathcal{R}(X) = \frac{1-\alpha^2}{\alpha}\sum_{n_1=0}^{\infty}\alpha^{2n_1}\exp[-\,(2n_1+1)\gamma X]$$
$$\times\left[1 - (1-\alpha^2)\sum_{n_2=0}^{\infty}\alpha^{2n_2}\exp(-\,2n_2\gamma X)\right].$$

This can be rewritten in the form of the single series

$$\mathcal{T}(X)\mathcal{R}(X) = \frac{1-\alpha^2}{\alpha}\sum_{n=0}^{\infty}\left[1 - (n+1)(1-\alpha^2)\right]\alpha^{2n}\exp[-\,(2n+1)\gamma X].$$

Again, replacing approximately $\langle \mathcal{T}(X)\mathcal{R}(X)\rangle = \int_0^\infty \mathrm{d}X\, \mathcal{T}(X)\mathcal{R}(X)$ by the Gaussian mean of $\Theta(X)\mathcal{T}(X)\mathcal{R}(X)$, we obtain the expression

$$\langle \mathcal{T}(X)\mathcal{R}(X)\rangle \approx \frac{1-\alpha^2}{2\alpha}\sum_{n=0}^{\infty}\left[1 - (n+1)(1-\alpha^2)\right]\alpha^{2n}$$
$$\times\, \mathrm{erfc}\left(\frac{(2n+1)\gamma\eta^2-\bar{X}}{\sqrt{2\eta^2}}\right)\exp\left(-\,(2n+1)\gamma\bar{X} + \frac{(2n+1)^2\gamma^2\eta^2}{2}\right). \tag{31}$$

In the end, let us note that if the condition $0 < \eta \ll \bar{X}$ is not fulfilled with a sufficient accuracy, the expressions on the right-hand sides of equations (29), (30) and (31) have to be multiplied, in the spirit of Section 2, by the factor

$$\left[1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{\bar{X}}{\sqrt{2\eta^2}}\right)\right]^{-1}.$$

Of course, this same comment is relevant in the case of formulae (22) and (23).

5 Results and discussion

We intend now to compare the deterministic results corresponding to $\eta_\sigma = 0$ (and then also $\eta = 0$) with the stochastic results that have been derived in Sections 3 and 4. The randomness of the function $\sigma(x)$ is defined by three parameters: $\bar{\sigma}$, η_σ and the correlation length ℓ of $\sigma(x)$.

If $\eta_\sigma \to +0$, our stochastic problem goes over into the deterministic one. (Since we have dealt with $\bar{\sigma}$ = const throughout the present paper, the deterministic case corresponds, of course, to the homogeneous sample without the fluctuations of σ when σ = const.) The mean value $\bar{X}$ of the optical length X of the rod is a dimensionless quantity proportional to the factual length L of the rod, $\bar{X} = \bar{\sigma} L$.

In order to present graphical presentation of our results, we deem it suitable to introduce the positive dimensionless parameter

$$\beta = \sqrt{\eta_\sigma \ell}\,\sqrt{\frac{\eta_\sigma}{\bar{\sigma}}} = \eta_\sigma\,\sqrt{\frac{\ell}{\bar{\sigma}}}\,. \tag{32}$$

If attention is paid on *man-made* media, the constants $\bar{\sigma}$, η_σ and ℓ depend on the technology used in the fabrication of these media. The constants $\bar{\sigma}$, η_σ and ℓ (and then also the parameter β) are *statistical* characteristics of the media. Namely, if we consider seemingly equal rods with the same values of $\bar{\sigma}$, η_σ and ℓ, and even if the length L of these rods is equal, the transmission and reflection coefficients, $\mathcal{T}$ and $\mathcal{R}$, respectively, may differ from one another from rod to rod. Therefore, our theory does not yield more than statistical predictions. According to our theory of the albedo problem, we can predict the mean values $\bar{\mathcal{T}}$ and $\bar{\mathcal{R}}$ but we can also grasp the statistics of how the values of $\mathcal{T}$ and $\mathcal{R}$ are dispersed around $\bar{\mathcal{T}}$ and $\bar{\mathcal{R}}$. The value of β may be more or less arbitrary, serving for the characterization of the random medium under consideration.

To estimate the value of the quantity η according to formula (17), we have to choose a typical shape of the autocorrelation function $W(|x|)$. (Actually, in man-made materials, the function $W(|x|)$ is technology-dependent.) For the sake of simplicity, let us take $W(|x|)$ in the exponential form, $W(|x|) = \eta_\sigma^2 \exp(-|x|/\ell)$. (Such a form of the autocorrelation function was used, for instance, in [6].) Since it is reasonable to assume that $\ell \ll L$, we have got the estimate

$$\eta^2 = \eta_\sigma^2 \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2\, \exp\left(-\,\frac{|x_2 - x_1|}{\ell}\right)$$
$$\approx\, \eta_\sigma^2 \int_0^L \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\,\frac{|x|}{\ell}\right) = 2\eta_\sigma^2 \ell\, L = 2\beta^2\,\bar{\sigma} L\,. \tag{33}$$

If $W(|x|)$ were chosen in a form differing from the simple exponential, or if the correlation length ℓ were defined with a factor different from (but of the same order of magnitude as) unity, formula (33) would still be valid, although with a factor different from 2. For the sake of simplicity, we will take formula (33) as approximately correct even in the case of small values of L.

Using the parameter β, we can rewrite condition (18) in the form

$$\frac{\eta}{\bar{X}} \,\approx\, \beta\sqrt{\frac{2}{\bar{\sigma} L}} \,\ll\, 1\,. \tag{34}$$

Thus, if $L \to \infty$, then $\eta/\bar{X} \to +0$. If condition (34) is not fulfilled, we have to use the corrective normalization factor

$$N(\bar{\sigma} L) = \left[1 \,-\, \frac{1}{2}\,\mathrm{erfc}\left(\frac{\bar{\sigma} L}{\sqrt{2\eta^2}}\right)\right]^{-1} = \left[1 \,-\, \frac{1}{2}\,\mathrm{erfc}\left(\frac{\sqrt{\bar{\sigma} L}}{2\beta}\right)\right]^{-1}$$

(cf. Section 2). With the parameter β, formulae (22), (23), (29), (30) and (31) read as follows:

$$\bar{\mathcal{T}} = N(\bar{\sigma}L)\,\frac{(1-\,\alpha^2)}{2}\,\sum_{n=0}^{\infty}\,\alpha^{2n}\,\mathrm{erfc}\left[\left((2n+1)\gamma\beta\,-\,\frac{1}{2\beta}\right)\sqrt{\bar{\sigma}L}\right]$$
$$\times\;\exp\left\{\left[-\;(2n+1)\gamma\,+\,(2n+1)^2\gamma^2\beta^2\,\right]\bar{\sigma}L\right\}, \tag{22$'$}$$

$$\bar{\mathcal{R}} = \frac{1}{\alpha}\left\{1\,-\,N(\bar{\sigma}L)\,\frac{1-\alpha^2}{2}\,\sum_{n=0}^{\infty}\,\alpha^{2n}\,\mathrm{erfc}\left[\left(2n\gamma\beta\,-\,\frac{1}{2\beta}\right)\sqrt{\bar{\sigma}L}\right]\right.$$
$$\times\;\exp\left[\left(-\;2n\gamma\,+\,4n^2\gamma^2\beta^2\right)\bar{\sigma}L\right], \tag{23$'$}$$

$$\langle\mathcal{T}^2\rangle = N(\bar{\sigma}L)\,\frac{(1-\alpha^2)^2}{2}\,\sum_{n=0}^{\infty}\,(n+1)\alpha^{2n}\,\mathrm{erfc}\left[\left(2(n+1)\gamma\beta\,-\,\frac{1}{2\beta}\right)\sqrt{\bar{\sigma}L}\right]$$
$$\times\;\exp\left[\left(-\;2(n+1)\gamma\,+\,4(n+1)^2\gamma^2\beta^2\right)\bar{\sigma}L\right], \tag{29$'$}$$

$$\langle\mathcal{R}^2\rangle =$$
$$\frac{1}{\alpha^2}\left[1\,+\,N(\bar{\sigma}L)\,\frac{1-\alpha^2}{2}\,\sum_{n=0}^{\infty}\,[(n+1)(1-\alpha^2)-2]\,\alpha^{2n}\,\mathrm{erfc}\left[\left(2n\gamma\beta\,-\,\frac{1}{2\beta}\right)\sqrt{\bar{\sigma}L}\right]\right.$$
$$\times\;\exp\left[\left(-\;2n\gamma\,+\,4n^2\gamma^2\beta^2\right)\bar{\sigma}L\right] \tag{30$'$}$$

and

$$\langle\mathcal{T}\mathcal{R}\rangle =$$
$$N(\bar{\sigma}L)\,\frac{1-\alpha^2}{2\alpha}\,\sum_{n=0}^{\infty}\,[1\,-\,(n+1)(1-\alpha^2)]\,\alpha^{2n}\,\mathrm{erfc}\left[\left((2n+1)\gamma\beta\,-\,\frac{1}{2\beta}\right)\sqrt{\bar{\sigma}L}\right]$$
$$\times\;\exp\left[\left(-\;(2n+1)\gamma\,+\,(2n+1)^2\gamma^2\beta^2\right)\bar{\sigma}L\right]. \tag{31$'$}$$

The value of c lies between zero and unity. To exemplify our calculations, we take $c = 0.9$. Then $\gamma = \sqrt{1-c} = 0.31623$ and $\alpha = (1-\gamma)/(1+\gamma) = 0.51949$. In Figs. $1,\ldots,5$, we show how some relevant variables of our stochastic albedo problem depend on the averaged optical length $\bar{\sigma}L$. The curves illustrating the stochastic problem correspond (in all the Figures) to calculations with $\beta = 0.3$. In Figs. 1 and 2, the dash lines correspond to $\beta = 0.3$ and the full lines to $\beta = 0$. (If $\beta = 0$, we may speak of the deterministic model. In this case, the function $\sigma(x)$ is reduced to a constant, for $\eta_\sigma = 0$.) The central dash lines in Figs. 1 and 2, corresponding, respectively, to the mean values $\bar{\mathcal{T}}$ and $\bar{\mathcal{R}}$, have not been plotted with the same sharpness as other lines, since they almost coincide with the full lines. This approximate coincidence proves that the dependence of the values of $\bar{\mathcal{T}}$ and $\bar{\mathcal{R}}$ on β is insignificant for small values of β. The upper and lower dash lines correspond to the values of $\bar{\mathcal{T}} \pm \eta_{\mathcal{T}}$ in Fig. 1, and to the values of $\bar{\mathcal{R}} \pm \eta_{\mathcal{R}}$ in Fig. 2. The distributions of the random values of $\mathcal{T}$ and $\mathcal{R}$ around the respective mean values $\bar{\mathcal{T}}$ and

$\bar{\mathcal{R}}$ may be approximated as Gaussians. Then we expect that almost 70 per cent of the stochastic values of $\mathcal{T}$ (obtained in casual measurements) will lie in the stripe between the outer dash lines in Fig. 1. We may say the same when discussing the statistics of potential measurements of the reflection coefficient $\mathcal{R}$: almost 70 per cent of the values of $\mathcal{R}$ are expected to lie in the stripe between the outer dash lines in Fig. 2. Asymptotically, for $L \to \infty$, the coefficient $\mathcal{R}$ tends to the value of α. (Clearly $\mathcal{R} \approx \bar{\mathcal{R}}$ if $L \to \infty$.)

In Fig. 3, we have plotted the r.m.s. deviations, $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{R}}$, of $\mathcal{T}$ and $\mathcal{R}$, respectively. If $L \to \infty$, both $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{R}}$ tend to zero. On the other hand, $\eta_{\mathcal{T}}$ and $\eta_{\mathcal{R}}$ do also tend to zero if $L \to +0$. (Obviously, if $L = 0$, the rod is absent; then there is no room for the stochasticity and $\mathcal{T} = 1$, $\mathcal{R} = 0$.) Since $\eta_{\mathcal{T}}$ ($\eta_{\mathcal{R}}$) is non-negative, the zero values of $\eta_{\mathcal{T}}$ ($\eta_{\mathcal{R}}$) at $L = 0$ and $L = \infty$ imply that there has to exist at least one maximum at a finite value of the rod length. (Our calculations have proved the existence of just one maximum.)

Figs. 4 and 5 illustrate the 'anticorrelation' between the stochastic variables $\mathcal{T}$ and $\mathcal{R}$. In Fig. 4, we have plotted the dependence of the off-diagonal element $C_{\mathcal{T}\mathcal{R}} = \langle(\mathcal{T} - \bar{\mathcal{T}})(\mathcal{R} - \bar{\mathcal{R}})\rangle$ on $\bar{\sigma}L$. Fig. 5 shows the dependence of the 'normalized' correlation function

$$F_{\mathcal{T}\mathcal{R}}(\bar{\sigma}L) = \frac{C_{\mathcal{T}\mathcal{R}}}{\eta_{\mathcal{T}}\eta_{\mathcal{R}}} \tag{35}$$

on $\bar{\sigma}L$. We speak of the anticorrelation since the averaged values of the product $(\mathcal{T} - \bar{\mathcal{T}})(\mathcal{R} - \bar{\mathcal{R}})$ are negative. This anticorrelation follows from formulae (10) and (11): if the value of the transmission coefficient $\mathcal{T}$ becomes increased as a consequence of varying the stochastic function $\sigma(x)$, this increase has to go hand in hand with a decrease in the value of the reflection coefficient $\mathcal{R}$.

In conclusion, let us point out that although we have confined ourselves to treating the one-dimensional stochastic 'rod model' in the present paper, our line of argument can be followed − although necessarily with a much more demanding mathematical effort − even in solving three-dimensional stochastic problems of the theory of the radiative transfer, provided that their formulation is appropriately simplified.

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